Lecture 3: Non-linear viscoelasticity

Start with in-homogeneous non-isothermal flow:

$$\rho C_p \left(\frac{\partial T}{\partial t} + v \cdot \nabla T \right) = k \, \nabla^2 T$$

What is the origin of the red term?

Can we simply use the LVE Maxwell model in the form:

$$\sigma + \lambda \frac{\partial}{\partial t} \sigma = \eta \dot{\gamma} = \eta \left[(\nabla v) + (\nabla v)^T \right]$$

DISCUSS

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Characteristic length L so that $\nabla = L^{-1}\hat{\nabla}$

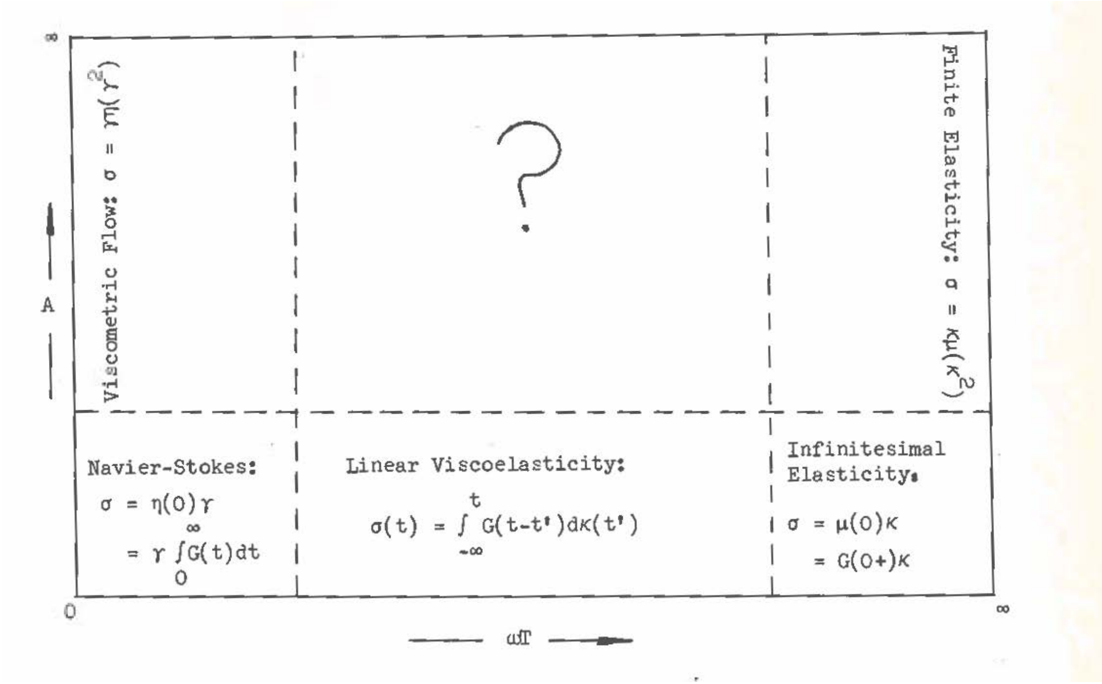
Characteristic velocity U so that $\mathrm{v} = U\hat{v}$

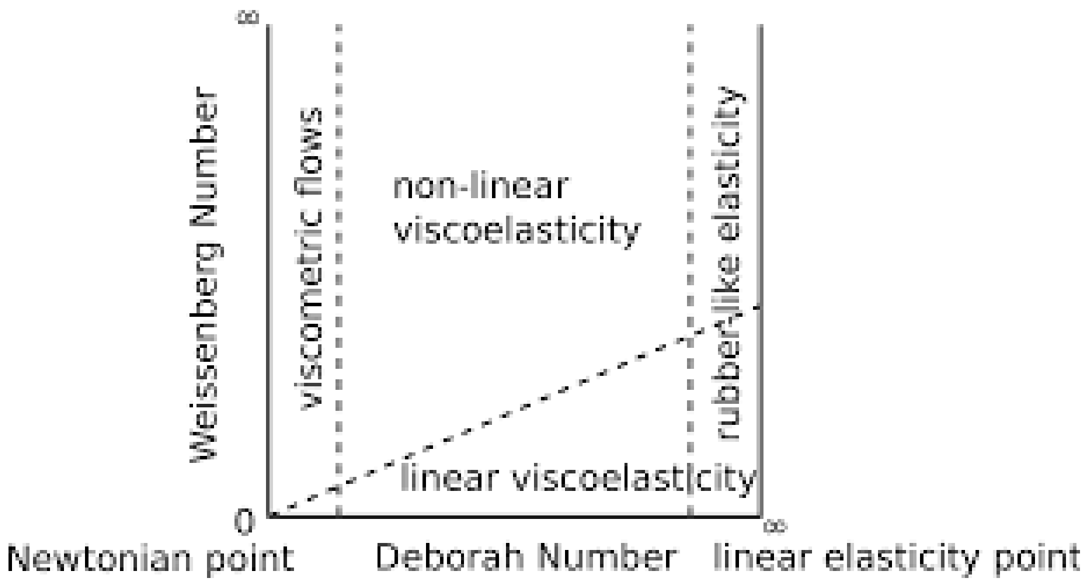
$$\sigma + \left(\frac{\partial}{\partial t/\lambda} + \frac{\lambda U}{L}\hat{v} \cdot \hat{\nabla}\right)\sigma = \eta\dot{\gamma}$$

Weissenberg number: $\frac{\lambda U}{L}$

Recall Deborah number: $De = \lambda\omega$

Pipkin diagram or Pipkin space:





A.C. Pipkin, Lectures on Viscoelasticity Theory, Springer, (1972)

But stress temperature is a scalar and stress is a tensor. The modified Maxwell eqn. Still not OK.

J.G. Oldroyd used ”convected coordinates” to derive two invariant time-derivatives.

The so-called ”upper-convected” time derivative is (DPL 7.1-1):

$$\sigma_{(1)} = \left(\frac{\partial}{\partial t} + v \cdot \nabla\right)\sigma - \{(\nabla v)^T \cdot \sigma + \sigma \cdot (\nabla v)\}$$

$$\sigma + \lambda\sigma_{(1)} = \eta\dot{\gamma} = \eta[(\nabla v) + (\nabla v)^T]$$

Exercise 3: Non-linear viscoelasticity

Problem 3.1-1: Steady shear flow of UCM

$$\begin{aligned} v_1 &= \dot{\gamma} x_2 \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned}$$

$$(\nabla v)_{ij} = \frac{\partial}{\partial x_i} v_j = \begin{bmatrix} 0 & \dot{\gamma} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

$$\begin{aligned} \sigma_{12} &= \eta(\dot{\gamma})\dot{\gamma} \\ \sigma_{11} - \sigma_{22} &= \Psi_1(\dot{\gamma})\dot{\gamma}^2 \\ \sigma_{22} - \sigma_{33} &= \Psi_2(\dot{\gamma})\dot{\gamma}^2 \end{aligned}$$

η, Ψ_1 and Ψ_2 define respectively the viscosity function, and the first and second normal stress coefficients.

Problem 3.1-2: Steady elongational flow of UCM:

$$\begin{aligned} v_1 &= -\frac{1}{2}\dot{\varepsilon}\, x_1 \\ v_2 &= -\frac{1}{2}\dot{\varepsilon}\, x_2 \\ v_3 &= \dot{\varepsilon}\, x_3 \end{aligned}$$

$$\begin{aligned} \sigma_{22} &= \sigma_{11} \\ \sigma_{33} - \sigma_{11} &= \eta_E(\dot{\varepsilon})\,\dot{\varepsilon} \end{aligned}$$

$\eta_E(\dot{\varepsilon})$ is the elongational or extensional viscosity function.

Problem 3.3: Start-up of steady shear flow of UCM

$$\begin{aligned} v_1 &= H(t)\,\dot{\gamma}x_2 \\ v_2 &= 0 \\ v_3 &= 0 \end{aligned}$$

Where the $H(t)$ is the Heaviside step function equal to 0 for negative arguments and equal to 1 for positive arguments.

Find the shear stress growth coefficient ($\eta^+(t,\dot{\gamma})$) and the first (Ψ_1^+) and the second (Ψ_2^+) normal stress growth coefficients defined by

$$\begin{aligned} \sigma_{12} &= \eta^+(t,\dot{\gamma})\dot{\gamma} \\ \sigma_{11}-\sigma_{22} &= \Psi_1^+(t,\dot{\gamma})\dot{\gamma}^2 \\ \sigma_{22}-\sigma_{33} &= \Psi_2^+(t,\dot{\gamma})\dot{\gamma}^2 \end{aligned}$$

Problem 3.4: Start-up of steady elongational flow of UCM:

$$\begin{aligned} v_1 &= -\frac{1}{2}H(t)\dot{\varepsilon}x_1 \\ v_2 &= -\frac{1}{2}H(t)\dot{\varepsilon}x_2 \\ v_3 &= H(t)\,\dot{\varepsilon}x_3 \end{aligned}$$

Find the tensile stress growth coefficient $\eta_E^+(t,\dot{\varepsilon})$ defined by

$$\sigma_{33} - \sigma_{11} = \eta_E^+(t,\dot{\varepsilon})\dot{\varepsilon}$$