4B.13 Tube Flow for the Truncated Power-Law Fluid

One way to estimate the importance of the zero-shear-rate viscosity on the volume flow rate in a tube is by using the simple truncated power-law model given in Table 4.5-1, Eqs. A and B. Let r_0 denote the radial position where $\dot{\gamma} = \dot{\gamma}_0$, that is where the viscosity changes from the zero-shear-rate value to the power-law function.

a. How must r_0 be restricted in order to use the truncated power-law model? Show that this corresponds to $(\eta_0 \dot{\gamma}_0 / \tau_R) < 1$ where τ_R is the wall shear stress in the tube.

b. Solve for the velocity field for $r \leq r_0$ and $r \geq r_0$. Match the results at $r = r_0$ in order to eliminate the last integration constant.

c. Show that the volume flow rate is given by Eq. H in Table 4.5-2.

d. Show that the result in part (c) reduces to the power-law result given in Eq. B of Table 4.2-1 or Eq. F of Table 4.5-2 when $(\eta_0 \dot{\gamma}_0 / \tau_R) \ll 1$. Note that the power-law consistency index m is related to the truncated power-law constants by $m = \eta_0 \dot{\gamma}_0^{1-n}$. Where does this come from?

4B.14 Variational Functional for Rod

Consider the functional in Eq. 4.3-16 and parallel the development in Eq. 4.3-3 *et seq.* In particular let $\overline{\Theta}(\xi) = \Theta(\xi) + \varepsilon\eta(\xi)$, where $\Theta(\xi)$ is the solution to Eqs. 4.3-13, 14 and 15, and $\eta(\xi)$ is an arbitrary function with $\eta(1) = 0$.

a. Show that when terms through second order in ε are retained one finds:

$$J\{\overline{\Theta}(\xi)\} = J\{\Theta(\xi)\} + \varepsilon^2 J\{\eta(\xi)\} \tag{4B.14-1}$$

b. Use the above to show that $\Theta(\xi)$ makes J a minimum.

4B.15 Development of Design Equation for Manifold of a "Coat-Hanger" Die (Power Law)[23]

Plastic sheeting can be made by extruding the molten polymer through a "coat-hanger" die made up of an entry tube, two manifolds, and a slit (see Fig. 4B.15). The manifold is a tube of circular cross section, whose radius $\bar{R}$ varies in the direction $\bar{z}$ of the manifold axis. Our object is to design the manifold (i.e., find $\bar{R}(\bar{z})$) so that the flow through the slit will be uniform; that is the volume flow rate of the slit must not vary in the x-direction.

The slit has a total width $2W$ and has a thickness $2B$. The volume rate of flow into the entry tube is $2Q_0$, with half of the fluid going into the left manifold and half into the right manifold.

a. Consider a width Δx of the slit. What is the volume flow rate through this portion of the slit?

b. Make a mass balance over a length $\Delta\bar{z}$ of the manifold tube and then let $\Delta\bar{z}$ go to zero to get the differential equation:

$$-\frac{d\bar{Q}}{d\bar{z}} = \frac{Q_0}{W} \cos\alpha \tag{4B.15-1}$$

where $\bar{Q}$ is the volume flow rate at $\bar{z}$. Draw a carefully labelled diagram to show how you derive this relation.

c. Let $\bar{p}(\bar{z})$ be the pressure as a function of $\bar{z}$ in the manifold. Let $p(z)$ be the pressure in the slit. Why is $p(-L(x)) \doteq \bar{p}(\bar{z})$? How are x and $\bar{z}$ related? How are W and $\bar{L}$ related?

[23] J. R. A. Pearson, *Trans. J. Plast. Inst.*, **32**, 239 (1964); J. R. A. Pearson, *Mechanics of Polymer Processing*, Elsevier Applied Science, New York (1985), §10.2. See also Z. Tadmor and C. G. Gogos, *Principles of Polymer Processing*, Wiley, New York (1979), pp. 545–551.

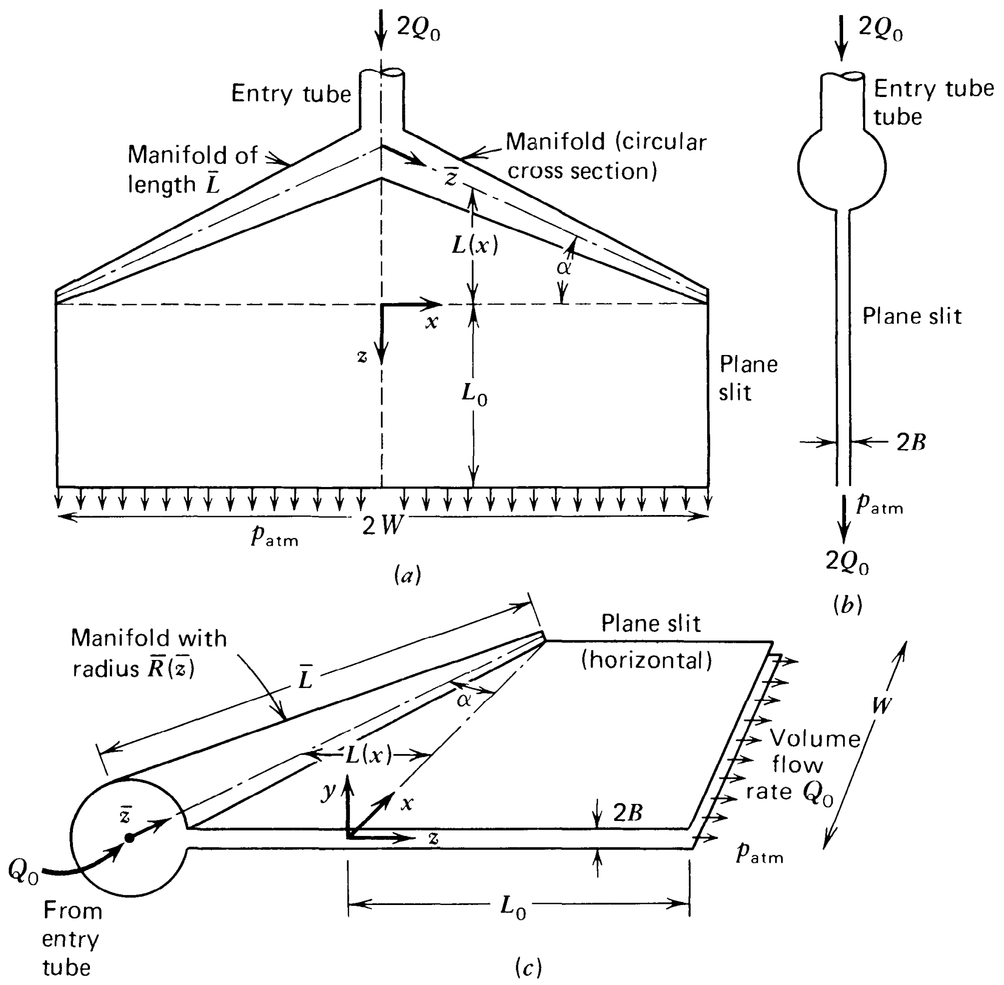

FIGURE 4B.15. The "coat-hanger" die, so named because the entry tube and manifold resemble a coat hanger. (*a*) Top view; (*b*) side view (looking in x-direction at a cut through the apparatus in the yz-plane); (*c*) one manifold and half of slit.

d. Adapt the power-law-fluid plane-slit-flow formula to a region of width Δx. Then use the results in **(c)** to obtain, finally,

$$-\frac{d\bar{p}}{d\bar{z}} = \left[\frac{Q_0}{W}\frac{[(1/n) + 2]}{2B^2}\right]^n \frac{m}{B} \sin \alpha \tag{4B.15-2}$$

e. Write the power-law analog of the Hagen–Poiseuille formula locally over a segment $d\bar{z}$ of the tube. This should contain $d\bar{p}/d\bar{z}$.

f. Combine the results of **(b)**, **(d)**, and **(e)** to get a differential equation for $\bar{R}$ as a function of $\bar{z}$. This should be of the form

$$\bar{R}^{(1/n)+2}\, d\bar{R}/d\bar{z} = \text{const} \tag{4B.15-3}$$

g. Integrate the differential equation to get $\bar{R}(\bar{z})$, the shape of the manifold. Take $\bar{R}$ to be zero at $\bar{z} = \bar{L}$ (just to make the problem a little simpler). Find ultimately that the desired design formula is

$$[\bar{R}(\bar{z})]^{3n+1} = \left[\frac{2}{\pi}\frac{[(1/n) + 3]}{[(1/n) + 2]}\right]^n \frac{2B^{2n+1}}{\sin \alpha} (W - x)^n$$

Show that if $n = 1/2$, then $\bar{R} \propto (W - x)^{1/5}$.

4B.15 Development of Design Equation for Manifold of a "Coat Hanger" Die (PLF) [RM]

a) For uniform flow: $\Delta Q = \frac{Q_o}{W}\Delta x$

b) $\bar{Q}$:= Flow rate into a section at a distance $\bar{z}$

Flow rate out of manifold section at $(\bar{z}+\Delta\bar{z})$

$= \bar{Q}+\Delta\bar{Q}$

$$\left\{\begin{matrix}\text{Flow into}\\ \text{manifold}\end{matrix}\right\}_{\bar{z}} = \left\{\begin{matrix}\text{Flow out}\\ \text{of manifold}\end{matrix}\right\}_{\bar{z}+\Delta\bar{z}} + \left\{\begin{matrix}\text{Flow into}\\ \text{plane slit}\end{matrix}\right\}_{\bar{z}+\Delta\bar{z}}$$

$$\bar{Q} = \bar{Q}+\Delta\bar{Q} + \frac{Q_o}{W}\Delta\bar{z}\cos\alpha$$

$$\therefore\ -\frac{\Delta\bar{Q}}{\Delta\bar{z}} = \frac{Q_o}{W}\cos\alpha \rightarrow -\frac{d\bar{Q}}{d\bar{z}} = \frac{Q_o}{W}\cos\alpha$$

c) The pt. $z=-L(x)$ is the pt. $\bar{z}$. Thus, if we denote the pressure distribution in the manifold by $\bar{p}(z)$ and that along the slit by $p(z)$:

$$p(-L(x)) = \bar{p}(\bar{z})\ ;\ \bar{z} = x/\cos\alpha\ ;\ \bar{L} = W/\cos\alpha$$

d) From Table 4.2-1, The pressure drop along plane slit: $-\frac{dp}{dz} = \left[\frac{Q_o[1/n+2]}{2B^2W}\right]^n\frac{m}{B}$

We assume that the pressure is uniform all across the slit. Therefore, the pressure drop along a distance L' (along $\bar{z}$) of the manifold is equal to that along a distance $L'\sin\alpha$ (along z) of the slit.

$$dz = d\bar{z}\,(\sin\alpha)$$

$$\therefore\; -\frac{d\bar{p}}{d\bar{z}} = \left[\frac{Q_0[1/n+2]}{W\;2B^2}\right]^n \frac{m}{B}\sin\alpha$$

e) From Ex. 4.2-1, we can write the equivalent of the Hagen-Poiseuille eqn. for the flow of a power-law fluid:

$$\bar{Q} = \frac{\pi\bar{R}^3}{1/n+3}\left[-\frac{d\bar{p}}{d\bar{z}}\frac{\bar{R}}{2m}\right]^{1/n} \quad \text{OR} \quad -\frac{d\bar{p}}{d\bar{z}} = \left[\frac{\bar{Q}\,[1/n+3]}{\pi\bar{R}^3}\right]^n \frac{2m}{\bar{R}}$$

f) $$\frac{d\bar{Q}}{d\bar{z}} = \pi\left[-\frac{d\bar{p}}{d\bar{z}}\cdot\frac{1}{2m}\right]^{1/n} \bar{R}^{(\frac{1}{n}+2)}\frac{d\bar{R}}{d\bar{z}}$$ by diff. above eqn.

Subst. $\frac{d\bar{p}}{d\bar{z}}$ from e) and $\frac{d\bar{Q}}{d\bar{z}}$ from b) into above eqn:

$$\bar{R}^{\frac{1}{n}+2}\frac{d\bar{R}}{d\bar{z}} = -\frac{Q_0}{\pi W}\cos\alpha\left[\frac{Q_0^n(\frac{1}{n}+2)^n}{(2B^2W)^n}\cdot\frac{m\sin\alpha}{B}\cdot\frac{1}{2m}\right]^{-1/n} \quad \text{or}$$

$$\bar{R}^{\frac{1}{n}+2}\frac{d\bar{R}}{d\bar{z}} = -\frac{2B^2\cos\alpha}{\pi(\frac{1}{n}+2)}\left[\frac{2B}{\sin\alpha}\right]^{1/n} = \text{constant}$$