

# **Polymer rheology, molecular theory and flow.**

Lecture series at Nanjing University, August 2018.

by

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The 8 lectures and associated 8 exercise sessions will be based on the two volume textbook Dynamics of Polymeric Liquids, "DPL" (Bird et al., Wiley 1987) with additional material. Active participation will be encouraged.

**Lecture 1: Basic concepts of flow:** The Navier-Stokes equation. Computational rheology with the generalized Newtonian fluid model. Shear flows and normal stresses. Application for polymer processing.

**Lecture 2: Linear viscoelasticity:** Relaxation functions. Continuous and discrete relaxation spectra. The Rouse spectrum. Interrelation between start-up, oscillation and creep. Non-isothermal properties and time-temperature shifting.

**Lecture 3: Non-linear viscoelasticity 1:** Material functions and empirical relations. Convected derivatives and the Oldroyd-B model. the Giesekus model, the Wiest model and the Rolie-Poly model.

**Lecture 4: Molecular theory 1:** The random walk and Gaussian springs. Diffusion equations and the molecular origin of stress. The Giesekus model, the Wiest model and the Rolie-Poly model..

**Lecture 5: Non-linear elasticity and viscoelasticity:** Deformation tensors, Gaussian network theory. Mooney-Rivlin plot. Integral models.

**Lecture 6: Molecular theory 2:** Reptation, the Doi-Edwards model, introduction to tube-based models with stretch..

**Lecture 7: Extensional rheology:** Techniques and relation to molecular structure.

**Lecture 8: Extensional rheology:** Information from neutron and x-ray scattering, fracture.

A temperature  $T$  is an example of a scalar. It has no direction.

A velocity  $\boldsymbol{v}$  is an example of a vector. It has a direction and coordinates  $v_i$  ( $i = 1,2,3$ ) in a rectangular coordinate system.

The temperature gradient  $\nabla T$  is another example of a vector. It has coordinates  $\partial/\partial x_i T$  in a rectangular coordinate system.

The velocity gradient  $\nabla \boldsymbol{v}$  is an example of a tensor. It has coordinates  $(\nabla \boldsymbol{v})_{ij} = (\partial/\partial x_i)v_j$ .

See Table A.7-1 and Table A.7-2 for definitions of differential operators.

Conservation of mass for incompressible fluids:

$$\nabla \cdot \boldsymbol{v} = 0 \quad (\text{DPL 1.1-5})$$

## Conservation of momentum (force balance)

$$\frac{\partial}{\partial t} \rho \mathbf{v} = -[\nabla \cdot \rho \mathbf{v} \mathbf{v}] - [\nabla \cdot \boldsymbol{\pi}] + \rho \mathbf{g}$$

$$\boldsymbol{\pi} = p \boldsymbol{\delta} + \boldsymbol{\tau}$$

$$\frac{\partial}{\partial t} \rho \mathbf{v} = -[\nabla \cdot \rho \mathbf{v} \mathbf{v}] - \nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}$$

So stress ( $\boldsymbol{\tau}$ ) in DPL is defined positive in compression and negative in tension.

In all other material in this course we use the symbol  $\boldsymbol{\sigma} = -\boldsymbol{\tau}$  for stress.

Then stress is positive in tension and negative in compression.

## Definition of stress:

Given a surface  $S$  with normal unit vector  $n$ .

The surface and normal vector divides space into two half spaces (positive side where the vector points into)

Then the force  $f$  exerted by material on the positive side on the material on the negative side is

$$f = \sigma \cdot n S$$

This is the defining equation for stress.

Incompressible Newtonian fluids:

$$\boldsymbol{\sigma} = \mu[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T] = \dot{\boldsymbol{\gamma}}$$

Inserted into momentum balance yields Navier-Stokes equation:

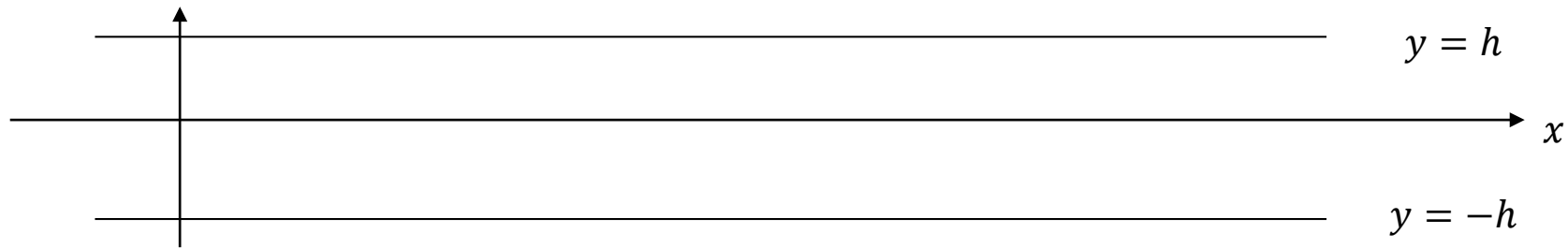
$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

DPL Table 1.2-1 (B)

Laminar flow between stationary parallel plane surfaces (DPL Example 1.3-1)

Two parallel planes separated by distance  $2h$  has a pressure gradient  $\partial/\partial x$ .

Find the volume rate of flow.



Assume that  $v_x = v_x(y)$  and  $v_y = v_z = 0$ .

$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

$$0 = -\frac{\partial}{\partial x} + \mu \frac{d^2 v_x}{dx^2}$$

$$v_x(-h) = v_x(h) = 0$$

$$Q = W \int_{-h}^h v_x dy = \frac{2Wh^3}{3\mu} \left( \frac{\partial p}{\partial x} \right)$$

## Laminar flow in circular tubes (DPL Example 1.3-2)

Given a horizontal tube of radius  $R$  and length  $L$  and pressure drop  $(p_0 - p_L)$ . Find the volume flow  $Q$ .

Assume that  $v_r = v_\theta = 0$ ,  $v_z = v_z(r)$  and  $p = p(z)$ .

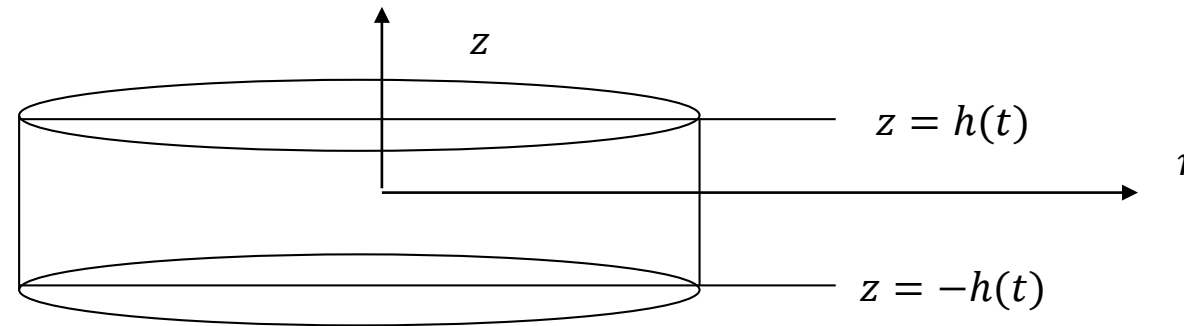
$$\rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

DPL Table B.2 Eqn. B.2-6:

$$0 = -\frac{dp}{dz} + \mu \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) \quad , \quad v_z(R) = 0 \quad , \quad v_z(0) \text{ finite.}$$

$$Q = \int_0^{2\pi} \int_0^R v_z r dr = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

## Squeezing flow between parallel disks (DPL Example 1.3-5)



Imagine a cylindrical surface at some distance  $r$  from the axis. The volume rate of flow is  $Q$ .

$$Q = \pi r^2 \left( -2 \frac{dh}{dt} \right) = \frac{2Wh^3}{3\mu} \left( \frac{\partial p}{\partial r} \right) = \frac{2(2\pi r)h^3}{3\mu} \left( \frac{\partial p}{\partial r} \right)$$

$$\frac{\partial p}{\partial r} = \frac{3\mu}{2} \left( -\frac{dh}{dt} \right) \frac{1}{h^3} r \quad p(R) = p_{atm}$$

$$F = 2\pi \int_{r=0}^R (p - p_{atm}) r dr = \frac{3\pi R^4 \mu}{8h^3} \left( -\frac{dh}{dt} \right) \quad \text{Stefan equation}$$

$$\boldsymbol{\sigma} = \eta(\dot{\gamma})\dot{\gamma} \quad \text{where}$$

$$\dot{\gamma} = \sqrt{\frac{1}{2} \sum_i \sum_j \dot{\gamma}_{ij} \dot{\gamma}_{ij}}$$

Carreau model:  $\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = [1 + (\lambda \dot{\gamma})^2]^{(n-1)/2}$  .

Power law model:  $\eta = \dot{\gamma}^{n-1}$

## Exercises:

- 1.1-1: Axial flow through tube: Go through Example 4.2-1 (verify Table 4.2-1 (B))
- 1.1-2: Axial flow in slit: Verify Table 4.2-1 (A).
- 1.2-3: Squeezing flow of power law fluid: Go through Example 4.2-7
- 1.2-4: Design equation for Coat-Hanger die: Solve problem 4B.15 in DPL.

TABLE A.7-1

Summary of Differential Operations Involving the  $\nabla$ -Operator in Rectangular Coordinates ( $x, y, z$ )

$$(\mathbf{V} \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad (\text{A})$$

$$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \quad (\text{B})$$

$$\begin{aligned} (\boldsymbol{\tau} : \nabla \mathbf{v}) &= \tau_{xx} \left( \frac{\partial v_x}{\partial x} \right) + \tau_{xy} \left( \frac{\partial v_x}{\partial y} \right) + \tau_{xz} \left( \frac{\partial v_x}{\partial z} \right) \\ &\quad + \tau_{yx} \left( \frac{\partial v_y}{\partial x} \right) + \tau_{yy} \left( \frac{\partial v_y}{\partial y} \right) + \tau_{yz} \left( \frac{\partial v_y}{\partial z} \right) \\ &\quad + \tau_{zx} \left( \frac{\partial v_z}{\partial x} \right) + \tau_{zy} \left( \frac{\partial v_z}{\partial y} \right) + \tau_{zz} \left( \frac{\partial v_z}{\partial z} \right) \end{aligned} \quad (\text{C})$$

$$[\nabla s]_x = \frac{\partial s}{\partial x} \quad (\text{D}) \quad [\mathbf{V} \times \mathbf{v}]_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \quad (\text{G})$$

$$[\nabla s]_y = \frac{\partial s}{\partial y} \quad (\text{E}) \quad [\mathbf{V} \times \mathbf{v}]_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \quad (\text{H})$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \quad (\text{F}) \quad [\mathbf{V} \times \mathbf{v}]_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (\text{I})$$

$$[\mathbf{V} \cdot \boldsymbol{\tau}]_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad (\text{J})$$

$$[\mathbf{V} \cdot \boldsymbol{\tau}]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \quad (\text{K})$$

$$[\mathbf{V} \cdot \boldsymbol{\tau}]_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \quad (\text{L})$$

$$[\nabla^2 \mathbf{v}]_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \quad (\text{M})$$

$$[\nabla^2 \mathbf{v}]_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \quad (\text{N})$$

$$[\nabla^2 \mathbf{v}]_z = \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \quad (\text{O})$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_x = v_x \left( \frac{\partial w_x}{\partial x} \right) + v_y \left( \frac{\partial w_x}{\partial y} \right) + v_z \left( \frac{\partial w_x}{\partial z} \right) \quad (\text{P})$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_y = v_x \left( \frac{\partial w_y}{\partial x} \right) + v_y \left( \frac{\partial w_y}{\partial y} \right) + v_z \left( \frac{\partial w_y}{\partial z} \right) \quad (\text{Q})$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_z = v_x \left( \frac{\partial w_z}{\partial x} \right) + v_y \left( \frac{\partial w_z}{\partial y} \right) + v_z \left( \frac{\partial w_z}{\partial z} \right) \quad (\text{R})$$

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**TABLE A.7-1** (Continued)

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$\{\mathbf{V}\mathbf{v}\}_{xx} = \frac{\partial v_x}{\partial x}$	(S)
$\{\mathbf{V}\mathbf{v}\}_{xy} = \frac{\partial v_y}{\partial x}$	(T)
$\{\mathbf{V}\mathbf{v}\}_{xz} = \frac{\partial v_z}{\partial x}$	(U)
$\{\mathbf{V}\mathbf{v}\}_{yx} = \frac{\partial v_x}{\partial y}$	(V)
$\{\mathbf{V}\mathbf{v}\}_{yy} = \frac{\partial v_y}{\partial y}$	(W)
$\{\mathbf{V}\mathbf{v}\}_{yz} = \frac{\partial v_z}{\partial y}$	(X)
$\{\mathbf{V}\mathbf{v}\}_{zx} = \frac{\partial v_x}{\partial z}$	(Y)
$\{\mathbf{V}\mathbf{v}\}_{zy} = \frac{\partial v_y}{\partial z}$	(Z)
$\{\mathbf{V}\mathbf{v}\}_{zz} = \frac{\partial v_z}{\partial z}$	(AA)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{xx} = (\mathbf{v} \cdot \mathbf{V})\tau_{xx}$	(BB)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{xy} = (\mathbf{v} \cdot \mathbf{V})\tau_{xy}$	(CC)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{xz} = (\mathbf{v} \cdot \mathbf{V})\tau_{xz}$	(DD)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{yx} = (\mathbf{v} \cdot \mathbf{V})\tau_{yx}$	(EE)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{yy} = (\mathbf{v} \cdot \mathbf{V})\tau_{yy}$	(FF)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{yz} = (\mathbf{v} \cdot \mathbf{V})\tau_{yz}$	(GG)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{zx} = (\mathbf{v} \cdot \mathbf{V})\tau_{zx}$	(HH)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{zy} = (\mathbf{v} \cdot \mathbf{V})\tau_{zy}$	(II)
$\{\mathbf{v} \cdot \mathbf{V}\boldsymbol{\tau}\}_{zz} = (\mathbf{v} \cdot \mathbf{V})\tau_{zz}$	(JJ)

where the operator  $(\mathbf{v} \cdot \mathbf{V}) = v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}$

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TABLE A.7-2

Summary of Differential Operations Involving the  $\nabla$ -Operator in Cylindrical Coordinates ( $r, \theta, z$ )

$$(\mathbf{V} \cdot \mathbf{v}) = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad (\text{A})$$

$$(\nabla^2 s) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial s}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \quad (\text{B})$$

$$\begin{aligned} (\boldsymbol{\tau} : \nabla \mathbf{v}) &= \tau_{rr} \left( \frac{\partial v_r}{\partial r} \right) + \tau_{r\theta} \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right) + \tau_{rz} \left( \frac{\partial v_r}{\partial z} \right) \\ &\quad + \tau_{\theta r} \left( \frac{\partial v_\theta}{\partial r} \right) + \tau_{\theta\theta} \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) + \tau_{\theta z} \left( \frac{\partial v_\theta}{\partial z} \right) \\ &\quad + \tau_{zr} \left( \frac{\partial v_z}{\partial r} \right) + \tau_{z\theta} \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right) + \tau_{zz} \left( \frac{\partial v_z}{\partial z} \right) \end{aligned} \quad (\text{C})$$

$$[\mathbf{V}s]_r = \frac{\partial s}{\partial r} \quad (\text{D}) \quad [\mathbf{V} \times \mathbf{v}]_r = \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad (\text{G})$$

$$[\mathbf{V}s]_\theta = \frac{1}{r} \frac{\partial s}{\partial \theta} \quad (\text{E}) \quad [\mathbf{V} \times \mathbf{v}]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \quad (\text{H})$$

$$[\mathbf{V}s]_z = \frac{\partial s}{\partial z} \quad (\text{F}) \quad [\mathbf{V} \times \mathbf{v}]_z = \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (\text{I})$$

$$[\mathbf{V} \cdot \boldsymbol{\tau}]_r = \frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta r} + \frac{\partial}{\partial z} \tau_{zr} - \frac{\tau_{\theta\theta}}{r} \quad (\text{J})$$

$$[\mathbf{V} \cdot \boldsymbol{\tau}]_\theta = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta\theta} + \frac{\partial}{\partial z} \tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} \quad (\text{K})$$

$$[\mathbf{V} \cdot \boldsymbol{\tau}]_z = \frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rz}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta z} + \frac{\partial}{\partial z} \tau_{zz} \quad (\text{L})$$

$$[\nabla^2 \mathbf{v}]_r = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \quad (\text{M})$$

$$[\nabla^2 \mathbf{v}]_\theta = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \quad (\text{N})$$

$$[\nabla^2 \mathbf{v}]_z = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \quad (\text{O})$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_r = v_r \left( \frac{\partial w_r}{\partial r} \right) + v_\theta \left( \frac{1}{r} \frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} \right) + v_z \left( \frac{\partial w_r}{\partial z} \right) \quad (\text{P})$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_\theta = v_r \left( \frac{\partial w_\theta}{\partial r} \right) + v_\theta \left( \frac{1}{r} \frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} \right) + v_z \left( \frac{\partial w_\theta}{\partial z} \right) \quad (\text{Q})$$

$$[\mathbf{v} \cdot \nabla \mathbf{w}]_z = v_r \left( \frac{\partial w_z}{\partial r} \right) + v_\theta \left( \frac{1}{r} \frac{\partial w_z}{\partial \theta} \right) + v_z \left( \frac{\partial w_z}{\partial z} \right) \quad (\text{R})$$

TABLE A.7-2 (Continued)

$\{\mathbf{Nv}\}_{rr} = \frac{\partial v_r}{\partial r}$	(S)
$\{\mathbf{Nv}\}_{r\theta} = \frac{\partial v_\theta}{\partial r}$	(T)
$\{\mathbf{Nv}\}_{rz} = \frac{\partial v_z}{\partial r}$	(U)
$\{\mathbf{Nv}\}_{\theta r} = \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}$	(V)
$\{\mathbf{Nv}\}_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$	(W)
$\{\mathbf{Nv}\}_{\theta z} = \frac{1}{r} \frac{\partial v_z}{\partial \theta}$	(X)
$\{\mathbf{Nv}\}_{zr} = \frac{\partial v_r}{\partial z}$	(Y)
$\{\mathbf{Nv}\}_{z\theta} = \frac{\partial v_\theta}{\partial z}$	(Z)
$\{\mathbf{Nv}\}_{zz} = \frac{\partial v_z}{\partial z}$	(AA)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{rr} = (\mathbf{v} \cdot \mathbf{N})\tau_{rr} - \frac{v_\theta}{r}(\tau_{r\theta} + \tau_{\theta r})$	(BB)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{r\theta} = (\mathbf{v} \cdot \mathbf{N})\tau_{r\theta} + \frac{v_\theta}{r}(\tau_{rr} - \tau_{\theta\theta})$	(CC)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{rz} = (\mathbf{v} \cdot \mathbf{N})\tau_{rz} - \frac{v_\theta}{r}\tau_{\theta z}$	(DD)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{\theta r} = (\mathbf{v} \cdot \mathbf{N})\tau_{\theta r} + \frac{v_\theta}{r}(\tau_{rr} - \tau_{\theta\theta})$	(EE)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{\theta\theta} = (\mathbf{v} \cdot \mathbf{N})\tau_{\theta\theta} + \frac{v_\theta}{r}(\tau_{r\theta} + \tau_{\theta r})$	(FF)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{\theta z} = (\mathbf{v} \cdot \mathbf{N})\tau_{\theta z} + \frac{v_\theta}{r}\tau_{rz}$	(GG)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{zr} = (\mathbf{v} \cdot \mathbf{N})\tau_{zr} - \frac{v_\theta}{r}\tau_{z\theta}$	(HH)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{z\theta} = (\mathbf{v} \cdot \mathbf{N})\tau_{z\theta} + \frac{v_\theta}{r}\tau_{zr}$	(II)
$\{\mathbf{v} \cdot \mathbf{N}\tau\}_{zz} = (\mathbf{v} \cdot \mathbf{N})\tau_{zz}$	(JJ)

where the operator  $(\mathbf{v} \cdot \mathbf{N}) = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$

TABLE B.1

The Equation of Motion<sup>a</sup> in Terms of  $\tau$

Rectangular Coordinates ( $x, y, z$ ):

$$\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial}{\partial x} v_x + v_y \frac{\partial}{\partial y} v_x + v_z \frac{\partial}{\partial z} v_x \right) = - \left[ \frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right] - \frac{\partial p}{\partial x} + \rho g_x \quad (\text{B.1-1})$$

$$\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial}{\partial x} v_y + v_y \frac{\partial}{\partial y} v_y + v_z \frac{\partial}{\partial z} v_y \right) = - \left[ \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right] - \frac{\partial p}{\partial y} + \rho g_y \quad (\text{B.1-2})$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial}{\partial x} v_z + v_y \frac{\partial}{\partial y} v_z + v_z \frac{\partial}{\partial z} v_z \right) = - \left[ \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \tau_{zz} \right] - \frac{\partial p}{\partial z} + \rho g_z \quad (\text{B.1-3})$$

Cylindrical Coordinates ( $r, \theta, z$ ):

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = - \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta r} + \frac{\partial}{\partial z} \tau_{zr} - \frac{\tau_{\theta\theta}}{r} \right] - \frac{\partial p}{\partial r} + \rho g_r \quad (\text{B.1-4})$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = - \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta\theta} + \frac{\partial}{\partial z} \tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} \right] - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \quad (\text{B.1-5})$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = - \left[ \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta z} + \frac{\partial}{\partial z} \tau_{zz} \right] - \frac{\partial p}{\partial z} + \rho g_z \quad (\text{B.1-6})$$

Spherical Coordinates ( $r, \theta, \phi$ ):

$$\begin{aligned} \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\ = - \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi r} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] - \frac{\partial p}{\partial r} + \rho g_r \end{aligned} \quad (\text{B.1-7})$$

$$\begin{aligned} \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \\ = - \left[ \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi\theta} + \frac{(\tau_{\theta r} - \tau_{r\theta}) - \tau_{\phi\phi} \cot \theta}{r} \right] - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \end{aligned} \quad (\text{B.1-8})$$

$$\begin{aligned} \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r} \cot \theta \right) \\ = - \left[ \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\phi} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi\phi} + \frac{(\tau_{\phi r} - \tau_{r\phi}) + \tau_{\phi\theta} \cot \theta}{r} \right] \\ - \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \rho g_\phi \end{aligned} \quad (\text{B.1-9})$$

<sup>a</sup> In these equations no assumption is made regarding the symmetry of  $\tau$ .

TABLE B.2

The Equation of Motion for a Newtonian Fluid with Constant Density ( $\rho$ ) and Constant Viscosity ( $\mu$ )

Rectangular Coordinates ( $x, y, z$ ):

$$\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = \mu \left[ \frac{\partial^2}{\partial x^2} v_x + \frac{\partial^2}{\partial y^2} v_x + \frac{\partial^2}{\partial z^2} v_x \right] - \frac{\partial p}{\partial x} + \rho g_x \quad (\text{B.2-1})$$

$$\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = \mu \left[ \frac{\partial^2}{\partial x^2} v_y + \frac{\partial^2}{\partial y^2} v_y + \frac{\partial^2}{\partial z^2} v_y \right] - \frac{\partial p}{\partial y} + \rho g_y \quad (\text{B.2-2})$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[ \frac{\partial^2}{\partial x^2} v_z + \frac{\partial^2}{\partial y^2} v_z + \frac{\partial^2}{\partial z^2} v_z \right] - \frac{\partial p}{\partial z} + \rho g_z \quad (\text{B.2-3})$$

Cylindrical Coordinates ( $r, \theta, z$ ):

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] - \frac{\partial p}{\partial r} + \rho g_r \quad (\text{B.2-4})$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \quad (\text{B.2-5})$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{\partial p}{\partial z} + \rho g_z \quad (\text{B.2-6})$$

Spherical Coordinates ( $r, \theta, \phi$ ):

$$\begin{aligned} \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\ = \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] - \frac{\partial p}{\partial r} + \rho g_r \end{aligned} \quad (\text{B.2-7})$$

$$\begin{aligned} \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \\ = \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] \\ - \frac{1}{r} \frac{\partial p}{\partial \theta} + \rho g_\theta \end{aligned} \quad (\text{B.2-8})$$

$$\begin{aligned} \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r} \cot \theta \right) \\ = \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) \right. \\ \left. + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \theta} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] - \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \rho g_\phi \end{aligned} \quad (\text{B.2-9})$$

TABLE B.3

The Rate-of-Strain Tensor  $\dot{\gamma} = \nabla v + (\nabla v)^\dagger$

Rectangular Coordinates ( $x, y, z$ ):

$$\dot{\gamma}_{xx} = 2 \frac{\partial v_x}{\partial x} \quad (\text{B.3-1})$$

$$\dot{\gamma}_{yy} = 2 \frac{\partial v_y}{\partial y} \quad (\text{B.3-2})$$

$$\dot{\gamma}_{zz} = 2 \frac{\partial v_z}{\partial z} \quad (\text{B.3-3})$$

$$\dot{\gamma}_{xy} = \dot{\gamma}_{yx} = \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \quad (\text{B.3-4})$$

$$\dot{\gamma}_{yz} = \dot{\gamma}_{zy} = \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \quad (\text{B.3-5})$$

$$\dot{\gamma}_{zx} = \dot{\gamma}_{xz} = \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \quad (\text{B.3-6})$$

Cylindrical Coordinates ( $r, \theta, z$ ):

$$\dot{\gamma}_{rr} = 2 \frac{\partial v_r}{\partial r} \quad (\text{B.3-7})$$

$$\dot{\gamma}_{\theta\theta} = 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \quad (\text{B.3-8})$$

$$\dot{\gamma}_{zz} = 2 \frac{\partial v_z}{\partial z} \quad (\text{B.3-9})$$

$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (\text{B.3-10})$$

$$\dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} = \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \quad (\text{B.3-11})$$

$$\dot{\gamma}_{zr} = \dot{\gamma}_{rz} = \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \quad (\text{B.3-12})$$

Spherical Coordinates ( $r, \theta, \phi$ ):

$$\dot{\gamma}_{rr} = 2 \frac{\partial v_r}{\partial r} \quad (\text{B.3-13})$$

$$\dot{\gamma}_{\theta\theta} = 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \quad (\text{B.3-14})$$

$$\dot{\gamma}_{\phi\phi} = 2 \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \quad (\text{B.3-15})$$

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**TABLE B.3** (Continued)

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$$\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} = r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (\text{B.3-16})$$

$$\dot{\gamma}_{\theta\phi} = \dot{\gamma}_{\phi\theta} = \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \quad (\text{B.3-17})$$

$$\dot{\gamma}_{\phi r} = \dot{\gamma}_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \quad (\text{B.3-18})$$


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**TABLE B.4**

**The Vorticity Tensor  $\omega = \nabla v - (\nabla v)^\dagger$**

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Rectangular Coordinates ( $x, y, z$ ):

$$\omega_{xy} = -\omega_{yx} = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (\text{B.4-1})$$

$$\omega_{yz} = -\omega_{zy} = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \quad (\text{B.4-2})$$

$$\omega_{zx} = -\omega_{xz} = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \quad (\text{B.4-3})$$

Cylindrical Coordinates ( $r, \theta, z$ ):

$$\omega_{r\theta} = -\omega_{\theta r} = \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (\text{B.4-4})$$

$$\omega_{\theta z} = -\omega_{z\theta} = \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad (\text{B.4-5})$$

$$\omega_{zr} = -\omega_{rz} = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \quad (\text{B.4-6})$$

Spherical Coordinates ( $r, \theta, \phi$ ):

$$\omega_{r\theta} = -\omega_{\theta r} = \frac{1}{r} \frac{\partial}{\partial r} (rv_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (\text{B.4-7})$$

$$\omega_{\theta\phi} = -\omega_{\phi\theta} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) - \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \quad (\text{B.4-8})$$

$$\omega_{\phi r} = -\omega_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{1}{r} \frac{\partial}{\partial r} (rv_\phi) \quad (\text{B.4-9})$$


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“substantial derivative” or the “material derivative,” since it describes time changes taking place at a particular element of the “substance” or “material.” It should be emphasized that there are *no* assumptions involved in going from Eqs. A, B, and C to Eqs. D, E, and F in Table 1.1-1.

The analytical solution to most of the problems in this book will begin with one of the three “boxed” equations above. Usually one will want them written out in component form in one of the standard orthogonal coordinate systems. For the reader’s convenience, in Appendix B we give the equation of motion in rectangular, cylindrical, and spherical coordinates; in Appendix A many  $\nabla$ -operations are tabulated in the same three coordinate systems and also in bipolar coordinates.

## §1.2 THE EQUATIONS OF CHANGE IN TERMS OF THE TRANSPORT PROPERTIES

The equations in §1.1 are valid for any fluid. In this section we specialize these results for “Newtonian fluids” to obtain the equations of classical hydrodynamics. Then in the next section we give several examples of solutions of classical hydrodynamics problems. In doing so we select those problems that pertain to viscometry and to which we shall refer in subsequent chapters. Additional examples may be found in textbooks on transport phenomena.<sup>1</sup>

For structurally simple fluids such as gases, gaseous mixtures, and low-molecular-weight liquids and their mixtures, it has been established experimentally that in a simple shearing motion  $v_x = v_x(y)$  the flux of  $x$ -momentum in the positive  $y$ -direction is given by “Newton’s law of viscosity,”  $\pi_{yx} = -\mu dv_x/dy$ , where  $\mu$  is the *viscosity* of the fluid. The appropriate generalization for arbitrary, time-dependent flows is:<sup>2,3</sup>

$$\begin{aligned}\pi &= p\delta + \tau \\ &= p\delta - \mu[\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger] + \left(\frac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})\delta\end{aligned}\quad (1.2-1)$$

where  $(\nabla\mathbf{v})^\dagger$  is the transpose of the dyadic  $\nabla\mathbf{v}$ , and  $\delta$  is the unit tensor. This expression reduces to the hydrostatic pressure when there are no velocity gradients; it contains all possible combinations of first derivatives of velocity components that are allowed if one assumes that the fluid is isotropic and that the momentum flux tensor is symmetric.<sup>2,3</sup> The symbol  $p$  represents the thermodynamic pressure,<sup>2</sup> which is related to the density  $\rho$  and the temperature  $T$  through a “thermodynamic equation of state,”  $p = p(\rho, T)$ ; that is, this is taken to be the same function that one uses in thermal equilibrium.

The tensor  $\tau$  is the part of the momentum flux tensor or stress tensor that is associated with the viscosity of the fluid. We shall usually refer to it simply as the “momentum flux tensor” or “stress tensor,” and use the terms “total momentum flux tensor” or “total stress tensor” for  $\pi$  when a distinction seems necessary. An equation that assigns a value to  $\tau$  is called a *constitutive equation*. Equation 1.2-1 is the constitutive equation for the Newtonian fluid.

Note that in generalizing Newton’s law of viscosity to arbitrary flows an additional transport property  $\kappa$ , the *dilatational viscosity*, arises. The dilatational viscosity is identically

<sup>1</sup> See, for example, R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York (1960), Chaps. 3, 4, 10, 11.

<sup>2</sup> L. Landau and E. M. Lifshitz, *Fluid Mechanics*, Addison-Wesley, Reading, MA (1959), pp. 47–48, 187–188.

<sup>3</sup> G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1967), Sects. 3.3 and 3.4.

zero for ideal, monatomic gases; for incompressible liquids  $(\nabla \cdot \mathbf{v}) = 0$ , and the term containing  $\kappa$  vanishes. Consequently the dilatational viscosity is of no importance in two key limiting cases, and no further mention will be made of it in this book.<sup>4</sup>

For all fluids the density  $\rho$  depends on the local thermodynamic state variables, such as pressure and temperature. However for liquids it is often a very good assumption to take the density to be constant. Such an idealized fluid is often called an “incompressible fluid”, and the *momentum flux tensor* simplifies to

$$\pi = p\delta + \tau = p\delta - \mu\dot{\gamma} \quad (1.2-2)$$

in which  $\dot{\gamma} = \nabla \mathbf{v} + (\nabla \mathbf{v})^t$  is the *rate-of-strain tensor* or *rate-of-deformation tensor*. When the incompressibility assumption is made, a problem arises as to the meaning of  $p$ . For example, for a pure, incompressible fluid at constant temperature a plot of  $p$  vs.  $\rho$  is a vertical straight line; that is, the function  $p(\rho)$  is many-valued. This poses no difficulty in solving hydrodynamic problems since only the gradient of  $p$  needs to be known. However, in connection with determining pressures at surfaces, an incompressible fluid theory can predict only pressure differences and not absolute values (unless, of course, the pressure on some bounding surface is specified through a boundary condition). For all discussions of Newtonian fluids in this book, Eq. 1.2-2 will be used for the momentum flux tensor; that is we will use the simple constitutive equation  $\tau = -\mu\dot{\gamma}$  for incompressible Newtonian fluids.

The *heat flux*  $\mathbf{q}$  for pure fluids and nondiffusing mixtures is given by “Fourier’s law of heat conduction”:

$$\mathbf{q} = -k\nabla T \quad (1.2-3)$$

in which  $k$  is the *thermal conductivity* and  $T$  is the temperature. For diffusing mixtures there are additional contributions to  $\mathbf{q}$ , but we do not discuss them here.<sup>1,2</sup>

Now that we have given the expressions in Eqs. 1.2-2 and 1.2-3 for the fluxes, let us turn to the equations of change, and particularly the *equations of change for incompressible Newtonian fluids*. These are listed in Table 1.2-1 and given in Appendix B in various coordinate systems. The equation of continuity was given earlier in Eq. 1.1-5. The equation of motion is obtained by substituting Eq. 1.2-2 into Eq. 1.1-8 and simplifying. The energy equation is obtained by first transforming Eq. 1.1-13 into an equation for temperature (by using standard thermodynamic transformations) and then inserting Fourier’s law (Eq. 1.2-3) for  $\mathbf{q}$ . This process is outlined in Problem 1B.8; the final equation contains  $\hat{C}_p$ , which is the heat capacity at constant pressure per unit mass.

The equations of change in Table 1.2-1 are easy to interpret physically:

**A and D:** The *equation of continuity* states that within a small fixed volume there can be no net rate of addition of mass.

**B and E:** The *equation of motion* states that the mass-times-acceleration of a fluid element equals the sum of the pressure, viscous, and gravitational forces acting on the element.

**C and F:** The *energy equation* states that the temperature of a fluid element changes as it moves along with the fluid because of heat conduction (the  $k$ -term) and heat production by viscous heating (the  $\mu$ -term).

<sup>4</sup> The dilatational viscosity of a liquid containing gas bubbles has been studied by G. K. Batchelor, *op. cit.*, pp. 253-255.

TABLE 1.2-1

Equations of Change for Newtonian Fluids with Constant  $\rho$ ,  $\mu$ , and  $k$ 

	Dimensional Forms	Dimensionless Forms <sup>a</sup>
Continuity	$(\mathbf{V} \cdot \mathbf{v}) = 0$ (A)	$(\mathbf{V}^* \cdot \mathbf{v}^*) = 0$ (D)
Motion <sup>b,c</sup>	$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$ (B)	$\text{Re} \frac{D\mathbf{v}^*}{Dt^*} = -\nabla^* p^* + \nabla^{*2} \mathbf{v}^* + (\text{Re}/\text{Fr}) \mathbf{g}/g$ (E)
Energy	$\rho \hat{C}_p \frac{DT}{Dt} = k \nabla^2 T + \frac{1}{2} \mu (\dot{\gamma} : \dot{\gamma})$ (C)	$\text{Pé} \frac{DT^*}{Dt^*} = \nabla^{*2} T^* + \frac{1}{2} \text{Br} (\dot{\gamma}^* : \dot{\gamma}^*)$ (F)

<sup>a</sup> The dimensionless forms are based on a reference length  $L$ , reference velocity  $V$ , a reference temperature  $T_0$ , and a reference temperature difference  $\Delta T_0$ . In terms of these  $\mathbf{v}^* = \mathbf{v}/V$ ,  $\nabla^* = L\nabla$ ,  $D/Dt^* = (L/V)D/Dt$ ,  $p^* = (L/\mu V)p$ ,  $T^* = (T - T_0)/\Delta T_0$  and  $\dot{\gamma}^* = (L/V)\dot{\gamma}$ . The Reynolds number  $\text{Re} = LV\rho/\mu$ , the Froude number  $\text{Fr} = V^2/gL$ , the Péclet number  $\text{Pé} = \rho \hat{C}_p LV/k$ , and the Brinkman number  $\text{Br} = \mu V^2/k\Delta T_0$  are groups that appear as a result of writing the equations in dimensionless form. Other dimensionless groups may enter through the boundary conditions.

<sup>b</sup> For incompressible fluids we may combine the pressure and the gravity terms as  $\nabla \mathcal{P} = \nabla p - \rho \mathbf{g}$  where  $\mathcal{P}$  is called the “modified pressure.” If the velocity is specified on the entire boundary, we can conclude that the gravitational acceleration has no effect on the velocity field. If forces are specified on part of the boundary, as in free surface flow, the modified pressure is not a useful concept. The nomenclature “modified pressure” was suggested by G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1967), p. 176.

<sup>c</sup> The substantial derivative is defined as  $D/Dt = \partial/\partial t + (\mathbf{v} \cdot \nabla)$ .

Before ending this section we consider in Example 1.2-1 a result that is useful when the force on an object is desired in a fluid mechanical analysis.

### EXAMPLE 1.2-1 Proof that Normal Stresses of Incompressible Newtonian Fluids Are Zero at Solid Surfaces

We consider a point  $P$  on a solid surface that is in contact with an incompressible Newtonian fluid. Use a rectangular coordinate system  $xyz$  whose origin is at  $P$  and whose  $z$ -axis is normal to the surface and points into the fluid, and show that  $\tau_{zz}|_{z=0} = 0$ .

**SOLUTION** The result follows from the definition of the normal stress component  $\tau_{zz}$  and the mass conservation equation:

$$\begin{aligned} \tau_{zz} \Big|_{z=0} &= -2\mu \frac{\partial v_z}{\partial z} \Big|_{z=0} \\ &= 2\mu \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \Big|_{z=0} = 0 \end{aligned} \quad (1.2-4)$$

In the last step we have used the “no slip” condition on the solid surface.

*Note:* The result does not apply on surfaces with slip (see Eq. 1.4-43). Also in later chapters we shall find that for polymeric liquids normal stresses are *not* zero at solid surfaces.

## §1.3 SOLUTION OF ISOTHERMAL FLOW PROBLEMS

In this section we illustrate the solution of fluid flow problems with the use of the equations for incompressible Newtonian fluids under isothermal conditions. Our starting

equations are then Eqs. A and B in Table 1.2-1. These are four partial differential equations for the four unknowns: pressure and three components of velocity. Extensive experimental testing has shown that these equations describe the incompressible flow of Newtonian fluids exactly. Analytical solutions are, however, not always easy to obtain. In fact the equations are among the most challenging and extensively studied equations of mathematical physics. As a consequence we have available numerous treatises giving analytical solutions and solution procedures for Newtonian fluid mechanics.<sup>1</sup>

In connection with the examples we introduce two important approximate procedures: (i) in Examples 1.3-3 and 4 we use the lubrication approximation, in which the flow in a nearly constant cross section is approximated locally as flow in an equivalent constant-cross-section geometry; (ii) in Example 1.3-5 we introduce the quasi-steady-state approximation, in which an unsteady flow with small inertial effects is treated as a succession of steady-state flows.

### EXAMPLE 1.3-1 Laminar Flow between Parallel Plane Surfaces

An incompressible Newtonian fluid is located in the space between two parallel plates that are separated by a distance  $B$  (see Fig. 1.3-1). The upper plate is moving in the  $+x$ -direction with a velocity  $V$ , thus contributing to the motion of the fluid. An additional contribution to the fluid motion is that due to a constant applied pressure gradient  $\partial p/\partial x$ . Find the velocity profile and the volume rate of flow. Assume that the flow is sufficiently slow that viscous heating is not important.

**SOLUTION** We postulate that in this system  $v_x = v_x(y)$ ,  $v_y = 0$ ,  $v_z = 0$ ,  $p = p(x, y)$ , and  $T = \text{constant}$ . We now apply these postulates to the equations of change in order to get the differential equations that describe the system. The equations of continuity and energy are clearly unimportant. The  $y$ -component of the equation of motion just gives the vertical pressure gradient, which is of no interest here. The  $x$ -component of the equation of motion becomes:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 v_x}{dy^2} \quad (1.3-1)$$

in which  $\partial p/\partial x$  was stated to be a constant. This equation has to be integrated with respect to  $y$  with the boundary conditions:

$$\text{At } y = 0: \quad v_x = 0 \quad (1.3-2)$$

$$\text{At } y = B: \quad v_x = V \quad (1.3-3)$$

The result is the velocity distribution:

$$v_x = V \left( \frac{y}{B} \right) - \frac{B^2}{2\mu} \frac{\partial p}{\partial x} \left[ \left( \frac{y}{B} \right) - \left( \frac{y}{B} \right)^2 \right] \quad (1.3-4)$$

<sup>1</sup> For example, H. Lamb, *Hydrodynamics*, Dover, New York (1945); L. M. Milne-Thompson, *Theoretical Hydrodynamics*, 5th ed., Macmillan, New York (1967); H. L. Dryden, F. D. Murnaghan, and M. Bateman, *Hydrodynamics*, Dover, New York (1956); R. Berker, "Intégrations des équations du mouvement d'un fluide visqueux incompressible," *Handbuch der Physik*, Vol. VIII/2, Springer, Heidelberg (1963), pp. 1-384; H. Schlichting, *Boundary Layer Theory*, McGraw-Hill, New York, 4th ed. (1960); G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge Univ. Press (1967); L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon, London (1959).

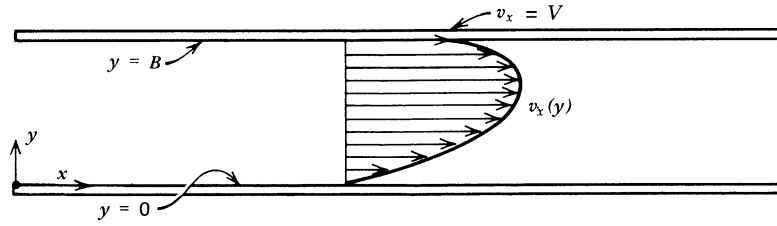


FIGURE 1.3-1. Flow between horizontal parallel planes with the upper plane moving and with an imposed pressure gradient in the flow direction.

The volume rate of flow  $Q$  for plates of width  $W$  is:

$$\begin{aligned}
 Q &= WB \langle v_x \rangle = \int_0^W \int_0^B v_x \, dy \, dz \\
 &= WB \int_0^1 v_x d\left(\frac{y}{B}\right) \\
 &= \frac{1}{2}WBV - \frac{WB^3}{12\mu} \frac{\partial p}{\partial x}
 \end{aligned} \tag{1.3-5}$$

Here the angular brackets  $\langle \rangle$  indicate an average over the cross section. This result contains the solution for the problem where the pressure gradient and the wall motion both tend to drive the fluid in the same direction and also the problem where the pressure gradient and wall motion oppose one another. Furthermore, the solution in Eq. 1.3-5 is seen to be a sum of the solutions to the two separate problems of wall driven flow and pressure driven flow. This superposition of solutions results, of course, from the linearity of the governing equations and boundary conditions. (*Note:* It is not usually possible to perform this superposition for non-Newtonian flow problems where the governing equations are nonlinear.)

### EXAMPLE 1.3-2 Laminar Flow in a Circular Tube

A fluid flows through a circular tube of radius  $R$  and length  $L$ . The tube makes an angle  $\chi$  with the vertical direction. The pressures at the tube ends at  $z = 0$  and  $z = L$  (see Fig. 1.3-2) are  $\mathcal{P}_0$  and  $\mathcal{P}_L$ , respectively. Find the steady-state velocity profile and the volume rate of flow, neglecting entrance and exit effects and assuming negligible viscous heating.

**SOLUTION** We postulate a solution of the form  $v_z = v_z(r)$ ,  $v_\theta = 0$ ,  $v_r = 0$ , and  $p = p(r, \theta, z)$ . The equation of continuity is satisfied identically, and only the  $z$ -component of the equation of motion is of interest:

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) + \rho g \cos \chi \tag{1.3-6}$$

This is to be solved with  $v_z = 0$  at  $r = R$  and  $v_z$  finite at  $r = 0$ .

Next we introduce<sup>2</sup> the “modified pressure”  $\mathcal{P} = p - \rho g z \cos \chi$  so that

$$\frac{\partial \mathcal{P}}{\partial z} = \mu \frac{1}{r} \frac{d}{dr} \left( r \frac{dv_z}{dr} \right) \tag{1.3-7}$$

<sup>2</sup> In general, for incompressible fluids  $\mathcal{P}$  is given by  $\mathcal{P} = p + \rho g h$ , where  $h$  is the distance *upward* (i.e., in the direction opposed to gravity) from some arbitrarily chosen reference plane. See also Table 1.2-1.

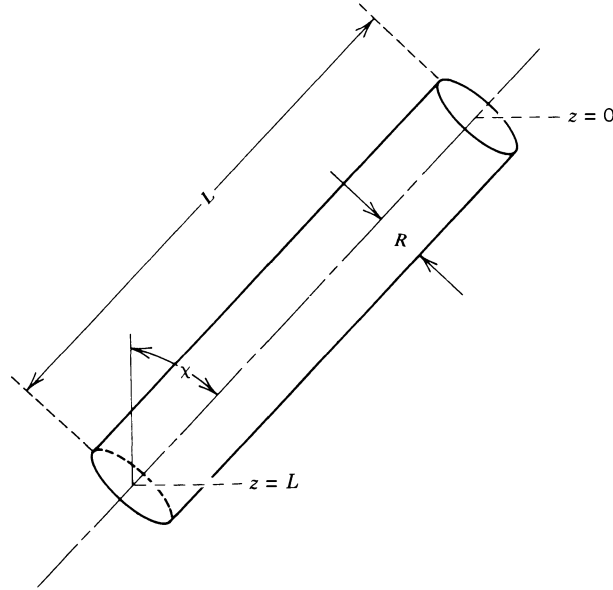


FIGURE 1.3-2. Flow through a circular tube that is inclined at an angle  $\chi$  to the vertical.

The right side is a function of  $r$  alone; let us call it  $F(r)$ . Then

$$\mathcal{P} = F(r)z + G(r, \theta) \quad (1.3-8)$$

Application of the boundary conditions at  $z = 0$  and  $z = L$  gives

$$\mathcal{P} = -\frac{\mathcal{P}_0 - \mathcal{P}_L}{L}z + \mathcal{P}_0 \quad (1.3-9)$$

When this is substituted into Eq. 1.3-7, the latter can be integrated; use of the boundary conditions that  $v_z = 0$  at  $r = R$  and that  $v_z$  is finite at  $r = 0$  then gives:

$$v_z = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R^2}{4\mu L} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad (1.3-10)$$

The volume rate of flow is then:

$$\begin{aligned} Q &= \pi R^2 \langle v_z \rangle = \int_0^{2\pi} \int_0^R v_z r \, dr \, d\theta \\ &= 2\pi R^2 \int_0^1 v_z \cdot \left( \frac{r}{R} \right) d\left( \frac{r}{R} \right) \\ &= \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R^4}{8\mu L} \end{aligned} \quad (1.3-11)$$

which is the famous result of Hagen and Poiseuille.<sup>3</sup> This relation (accompanied by additional information about end corrections) is the basic equation needed to determine viscosity from tube flow data. It is valid for  $Re < 2100$ , where the Reynolds number  $Re = 2R\langle v_z \rangle\rho/\mu = 2Q\rho/\pi\mu R$ , the angular brackets indicating an average over the cross section. For  $Re > 2100$ , the flow will usually be turbulent.

<sup>3</sup> G. Hagen, *Ann. Phys. Chem.*, **46**, 423-442 (1839); J. L. Poiseuille, *Comptes Rendus*, **11**, 961, 1041 (1840); **12**, 112 (1841).

**EXAMPLE 1.3-3** Flow in a Slightly Tapered Tube

An incompressible Newtonian fluid is flowing through the horizontal tapered tube shown in Fig. 1.3-3. Show that the analysis in Example 1.3-2 for a straight tube can be applied locally to this flow, and then use that result to obtain a relationship between the volume flow rate  $Q$  and the overall pressure drop ( $\mathcal{P}_0 - \mathcal{P}_L$ )  $\equiv \Delta\mathcal{P}$ .

**SOLUTION** The taper of the tube will require a flow in the radial direction and an acceleration in the axial direction, but it is reasonable to assume that the flow will maintain axial symmetry. Therefore we assume  $v_z = v_z(r, z)$ ,  $v_r = v_r(r, z)$ , and  $v_\theta = 0$ . The continuity equation and the  $r$ - and  $z$ -components of the Navier-Stokes equations are then (cf. Eqs. B.2-4 and 6)

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{\partial v_z}{\partial z} = 0 \quad (1.3-12)$$

$$\rho \left( v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (rv_r) \right) + \frac{\partial^2 v_r}{\partial z^2} \right] - \frac{\partial \mathcal{P}}{\partial r} \quad (1.3-13)$$

$$\rho \left( v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{\partial \mathcal{P}}{\partial z} \quad (1.3-14)$$

The nonuniform geometry has changed the linear problem in Eq. 1.3-6 to a nonlinear one; Eqs. 1.3-12 through 14 are difficult to solve in general. However we can take advantage of the fact that the geometry changes slowly to show that these equations are dominated by only a few terms, and that if small terms are neglected, the problem is easily solved.

To see which terms must be kept and which can be neglected, we perform an *order of magnitude analysis* on Eqs. 1.3-12 through 14. First we estimate the sizes of the velocities. The axial velocity is determined by the volume flow rate and must be of order  $(Q/\pi R_L^2) \equiv V$ . We write this as

$$v_z \sim O(V) = O(Q/\pi R_L^2) \quad (1.3-15)$$

The size of  $v_r$  is dictated by the continuity equation. In this equation the  $\partial v_z/\partial z$  term has magnitude

$$\begin{aligned} \frac{\partial v_z}{\partial z} &\sim O\left(\left(\frac{Q}{\pi R_L^2} - \frac{Q}{\pi R_0^2}\right)/L\right) \\ &= O(V(1 - (R_L/R_0)^2)/L) \end{aligned} \quad (1.3-16)$$

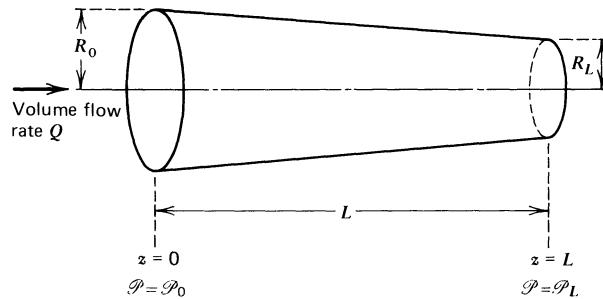


FIGURE 1.3-3. Tapered tube geometry analyzed in Example 1.3-3. The tube radius changes gradually from  $R_0$  to  $R_L$  over a distance  $L$ . It is not necessary that  $dR/dz$  be constant as shown here, but it must be small for the analysis to hold.

Clearly we expect this derivative to be small since  $(R_0 - R_L)/L \ll 1$ . If we let  $U$  denote the order of  $v_r$ , then the other contribution to the continuity equation is

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) \sim O\left(\frac{U}{R_L}\right) \quad (1.3-17)$$

The radial derivative is approximated by estimating that  $v_r$  can vary from  $U$  to 0 over a distance of  $R_L$ . From the equation of continuity in Eq. 1.3-12 we may now find the relative sizes of the radial and axial velocities:

$$U = V \left(\frac{R_L}{L}\right) \left[1 - \left(\frac{R_L}{R_0}\right)^2\right] \quad (1.3-18)$$

We now look at the sizes of the contributions to the equation of motion. The viscous terms have orders:

$$\mu \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (rv_r)\right) \sim O\left(\mu \frac{V}{R_L L} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-19)$$

$$\mu \frac{\partial^2 v_r}{\partial z^2} \sim O\left(\mu \frac{V}{L^3} R_L \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-20)$$

$$\mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r}\right) \sim O\left(\mu \frac{V}{R_L^2}\right) \quad (1.3-21)$$

$$\mu \frac{\partial^2 v_z}{\partial z^2} \sim O\left(\mu \frac{V}{L^2} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-22)$$

From these results it is seen that the dashed underlined terms in Eqs. 1.3-13 and 14 are smaller than the unmarked viscous term by at least a factor of  $(R_L/L)[1 - (R_L/R_0)^2] \ll 1$ .

Similarly we estimate the sizes of the inertial terms:

$$\rho v_r \frac{\partial v_r}{\partial r} \sim O\left(\rho \frac{V^2 R_L}{L^2} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]^2\right) \quad (1.3-23)$$

$$\rho v_z \frac{\partial v_r}{\partial z} \sim O\left(\rho V^2 \frac{R_L}{L^2} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-24)$$

$$\rho v_r \frac{\partial v_z}{\partial r} \sim O\left(\rho \frac{V^2}{L} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-25)$$

$$\rho v_z \frac{\partial v_z}{\partial z} \sim O\left(\rho \frac{V^2}{L} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-26)$$

When the largest of these (Eqs. 1.3-25 and 26) are compared with the largest viscous term, Eq. 1.3-21, we see that all are negligible provided

$$\frac{\rho V R_L}{\mu} \left[\frac{R_L}{L} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right] = \frac{1}{2} \text{Re} \left[\frac{R_L}{L} \left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right] \ll 1 \quad (1.3-27)$$

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It is not necessary in this problem that the Reynolds number be small for the inertial terms to be negligible, since it is multiplied by a small geometric factor associated with the taper of the tube.

The pressure gradient terms are the last to be estimated. In order for the  $r$  and  $z$  components of the Navier-Stokes equations to be satisfied, the pressure gradient terms must be of the same size as the largest viscous term in each. Thus

$$\frac{\partial \mathcal{P}/\partial r}{\partial \mathcal{P}/\partial z} \sim O\left(\left(\frac{R_L}{L}\right)\left[1 - \left(\frac{R_L}{R_0}\right)^2\right]\right) \quad (1.3-28)$$

It can now be seen that if

$$\left(\frac{R_L}{L}\right)\left[1 - \left(\frac{R_L}{R_0}\right)^2\right] \ll 1 \quad (1.3-29)$$

the equation of motion is well approximated by Eq. 1.3-7, which describes flow in the straight tube. Integrating Eq. 1.3-7 with respect to  $r$  leads as before to

$$Q = \frac{\pi[R(z)]^4}{8\mu} \left(-\frac{d\mathcal{P}}{dz}\right) \quad (1.3-30)$$

Notice that we have had to hold  $z$  constant during this integration, because the boundary conditions vary with  $z$ . Although this result could easily have been adapted from Eq. 1.3-11, the above analysis serves to organize the approximation procedure and to document the limit of validity (Eq. 1.3-29) of the result. The process that we have illustrated here of adapting locally the results for a uniform geometry to a slowly varying geometry is known as the *lubrication approximation*<sup>4</sup>. It is a very powerful technique for estimating the flow rate vs. pressure drop relation in many complex flows.

Finally we obtain the relation between pressure drop ( $\mathcal{P}_0 - \mathcal{P}_L$ ) and volume flow rate. Rather than use  $z$  as the independent variable in Eq. 1.3-30, we can use  $R = R_0 + (R_L - R_0)(z/L)$ :

$$Q = \frac{\pi R^4}{8\mu} \left(-\frac{d\mathcal{P}}{dR}\right) \left(\frac{R_L - R_0}{L}\right) \quad (1.3-31)$$

But  $Q$  is constant for all  $z$  (and hence all  $R$ ). Therefore the differential equation for  $\mathcal{P}$  as a function of  $R$  can be integrated to give:

$$\begin{aligned} Q &= \frac{3\pi(\mathcal{P}_0 - \mathcal{P}_L)}{8\mu} \frac{(R_0 - R_L)}{L} \frac{(R_0^{-3} - R_L^{-3})}{(R_L^{-3} - R_0^{-3})} \\ &= \frac{\pi(\mathcal{P}_0 - \mathcal{P}_L)R_0^4}{8\mu L} \left[1 - \frac{1 + (R_L/R_0) + (R_L/R_0)^2 - 3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2}\right] \end{aligned} \quad (1.3-32)$$

Hence the final result may be expressed as the Hagen-Poiseuille result multiplied by a correction factor.

### EXAMPLE 1.3-4 The Cone-and-Plate Viscometer

The cone-and-plate geometry shown in Fig. 1.3-4 is a standard experimental arrangement for the measurement of viscosity (and other rheological properties, as we shall see later). Obtain the analytical relations needed to interpret the instrumental data:

<sup>4</sup> J. R. A. Pearson, *Mechanics of Polymer Processing*, Elsevier Applied Science Publishers, London (1985), pp. 165-177; R. I. Tanner, *Engineering Rheology*, Oxford Univ. Press (1985), pp. 228-236.

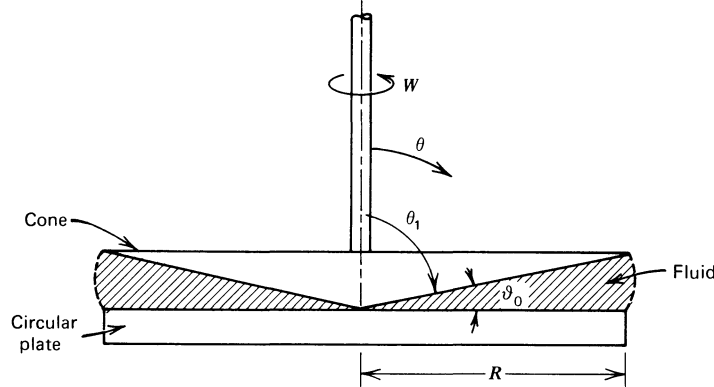


FIGURE 1.3-4. Cone-and-plate instrument; the angle  $\vartheta_0$  is usually between 0.5 and 8 degrees in commercial instruments.

- A relation between the angular velocity  $W$  and the  $\theta\phi$ -component of the rate-of-strain tensor  $\dot{\gamma}$  in the gap.
- A relationship between the torque  $\mathcal{T}$  and the  $\theta\phi$ -component of the stress tensor  $\tau$  in the gap.
- A relation giving the viscosity  $\mu$  in terms of  $W$  and  $\mathcal{T}$ .

**SOLUTION** The simplest analysis of this experiment makes use of the fact that the angle  $\vartheta_0$  is so small that a lubrication approximation may be applied to the flow in the gap. In this example, however, we do not perform an order of magnitude analysis as in the previous example, but rather apply the lubrication approximation intuitively by regarding the flow to be locally that between parallel plates.

(a) The velocity  $v_\phi$  at a radius  $r$  can be approximated by adapting Eq. 1.3-4 with  $\partial p/\partial x = 0$  and with  $v_x$  replaced by  $v_\phi$ . For the cone-and-plate system, at a distance  $r$  from the cone apex, the velocity of the cone will be  $Wr$  (this corresponds to  $V$  in Eq. 1.3-4), and the plate-cone separation will be given by  $r \sin \vartheta_0 \doteq r\vartheta_0$  (this corresponds to  $B$  in Eq. 1.3-4). Hence the velocity profile will, to a good approximation, be

$$v_\phi = Wr \left( \frac{(\pi/2) - \theta}{(\pi/2) - \theta_1} \right) \quad (1.3-33)$$

The  $\theta\phi$ -component of the  $\dot{\gamma}$ -tensor is then (cf. Eq. B.3-17):

$$\dot{\gamma}_{\theta\phi} = \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) \doteq \frac{1}{r} \frac{\partial}{\partial \theta} v_\phi = -\frac{W}{\vartheta_0} \quad (1.3-34)$$

The approximation made here is that  $\theta$  is so close to  $\pi/2$  that  $\sin \theta$  can be taken to be unity; this should be an excellent approximation. We see from Eq. 1.3-34 that  $\dot{\gamma}_{\theta\phi}$  is constant throughout the cone-plate gap. This is one reason why this geometry is useful for macromolecular fluids where the viscosity depends on the shear rate.

(b) The torque required to maintain the motion will be obtained by integrating the product of the lever arm  $r$  and the force  $\tau_{\theta\phi}|_{\theta=\pi/2} \cdot r dr d\phi$  over the surface of the plate

$$\mathcal{T} = \int_0^{2\pi} \int_0^R \tau_{\theta\phi} \Big|_{\theta=\pi/2} r^2 dr d\phi \quad (1.3-35)$$

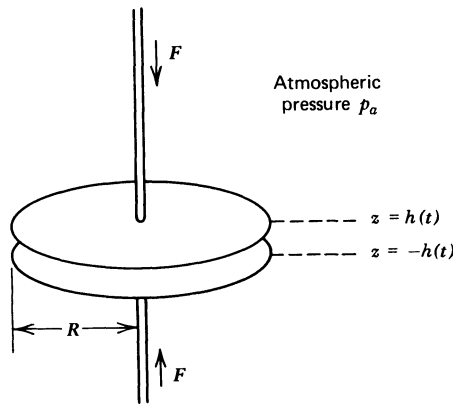


FIGURE 1.3-5. Squeezing flow between parallel disks each with radius  $R$ . The initial disk separation is  $2h_0$ .

Since  $\dot{\gamma}_{\theta\phi}$  is constant throughout the gap,  $\tau_{\theta\phi}$  will also be constant. Hence the integration is easily performed and one gets

$$\tau_{\theta\phi} = \frac{3\mathcal{F}}{2\pi R^3} \quad (1.3-36)$$

(c) Since  $\tau = -\mu\dot{\gamma}$  for a Newtonian fluid, the  $\theta\phi$ -component of this equation, combined with Eqs. 1.3-34 and 36, gives

$$\mu = \frac{\tau_{\theta\phi}}{-\dot{\gamma}_{\theta\phi}} = \frac{3\mathcal{F}\vartheta_0}{2\pi R^3 W} \quad (1.3-37)$$

This gives the viscosity of the fluid in the gap in terms of the geometrical quantities  $R$  and  $\vartheta_0$  and the measured values of torque  $\mathcal{F}$  and angular velocity  $W$ .

More complete analyses can be found elsewhere.<sup>5</sup> The treatment of this flow for non-Newtonian fluids is found in §10.2.

### EXAMPLE 1.3-5 Squeezing Flow between Two Parallel Disks<sup>6</sup>

A fluid is placed in the gap between two parallel disks separated by a distance  $2h_0$ . The fluid completely fills the gap. A constant force  $F$  is applied to each disk as shown in Fig. 1.3-5. It is desired to obtain an expression for the change in gap separation with time. A “quasi-steady state” solution will be used; that is, at any time  $t$  the radial flow problem will be treated as a steady-state hydrodynamic problem. This means that the inertial terms in the equation of motion are neglected in the first approximation. In addition gravity is assumed negligible.

**SOLUTION** We introduce a cylindrical coordinate system with  $z$ -axis coinciding with the symmetry axis of the flow and with the two disks defined by  $z = -h(t)$  and  $z = h(t)$ . Since  $R \gg h$  the flow will primarily be in the  $r$ -direction so that  $v_z \ll v_r$  and also  $(\partial v_r / \partial r) \ll (\partial v_r / \partial z)$ . Consistent with the

<sup>5</sup> R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *op. cit.*, p. 119; S. Oka in F. R. Eirich, ed., *Rheology*, Vol. 3, Academic Press, New York (1960), Chapt. 2, pp. 61-62; K. Walters, *Rheometry*, Chapman and Hall, London (1975), Chapt. 4.

<sup>6</sup> J. Stefan, *Sitzungber. K. Akad. Wiss. Math. Natur. Wien*, **69**, Part 2, 713-735 (1874); see also L. Landau and E. M. Lifshitz, *op. cit.*, pp. 70-71.

quasi-steady-state approximation, we take  $\rho \partial v_r / \partial t \ll \mu \partial^2 v_r / \partial z^2$ . Consequently the equation of continuity and the  $r$ - and  $z$ - components of the equation of motion are well approximated by

$$\text{Continuity: } \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{\partial v_z}{\partial z} = 0 \quad (1.3-38)$$

$$\text{Motion (r): } 0 = -\frac{\partial p}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \quad (1.3-39)$$

$$\text{Motion (z): } 0 = -\frac{\partial p}{\partial z} \quad (1.3-40)$$

We seek a solution for the velocity field in the form  $v_r = v_r(r, z, t)$  and  $v_z = v_z(z, t)$ . The continuity equation then demands that  $v_r = r f(z, t)$ . Furthermore, the equations of motion show that  $p$  must have the form  $p = p_0 + p_2 r^2$ , where  $p_0$  and  $p_2$  are constants to be determined. With these simplifications Eq. 1.3-40 is satisfied and Eqs. 1.3-38 and 39 give

$$2f + \frac{\partial v_z}{\partial z} = 0 \quad (1.3-41)$$

$$-2p_2 + \mu \frac{\partial^2 f}{\partial z^2} = 0 \quad (1.3-42)$$

The following boundary conditions are to be satisfied:

$$\partial f / \partial z = 0, \quad \text{at } z = 0; \quad f = 0, \quad \text{at } z = h \quad (1.3-43,44)$$

$$v_z = 0, \quad \text{at } z = 0; \quad v_z = \dot{h}, \quad \text{at } z = h \quad (1.3-45,46)$$

$$p = p_a, \quad \text{at } r = R \quad (1.3-47)$$

Here  $\dot{h}$  stands for  $dh/dt$ . These 5 conditions suffice to determine  $p_0$ ,  $p_2$ , and the 3 constants of integration of Eqs. 1.3-41 and 42. The results are

$$v_r = r f = \frac{3(-\dot{h})}{4h} r \left[ 1 - \left( \frac{z}{h} \right)^2 \right] \quad (1.3-48)$$

$$v_z = \frac{3}{2} \dot{h} \left[ \left( \frac{z}{h} \right) - \frac{1}{3} \left( \frac{z}{h} \right)^3 \right] \quad (1.3-49)$$

$$p - p_a = \frac{3(-\dot{h})\mu R^2}{4h^3} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \quad (1.3-50)$$

To calculate the force on one plate all we need is the pressure distribution in Eq. 1.3-50, since we know from Example 1.2-1 that  $\tau_{zz} = 0$  on the plates. Consequently we find

$$\begin{aligned} F &= \int_0^{2\pi} \int_0^R (p - p_a + \tau_{zz}) \Big|_{z=h} r \, dr \, d\theta \\ &= 2\pi R^2 \int_0^1 (p - p_a) \left( \frac{r}{R} \right) d\left( \frac{r}{R} \right) \\ &= \frac{3\pi R^4 \mu (-\dot{h})}{8h^3} \end{aligned} \quad (1.3-51)$$

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This is the *Stefan equation*, which shows how much force  $F(t)$  must be applied in order to maintain the disk motion  $h(t)$ .

If we now ask what the disk motion will be for a *constant applied force*  $F$ , we have to solve the differential equation for  $h(t)$  in Eq. 1.3-51 to give

$$\frac{1}{h^2} - \frac{1}{h_0^2} = \frac{16Ft}{3\pi R^4 \mu} \quad (1.3-52)$$

This gives the disk separation as a function of the elapsed time, when inertial effects may be neglected.

It is possible to correct Eq. 1.3-51 for small inertial effects by means of a perturbation approach. Then the creeping flow solution in Eqs. 1.3-48 and 49 is used to evaluate the inertial term  $\rho[\partial v_r/\partial t + v_r(\partial v_r/\partial r) + v_z(\partial v_r/\partial z)]$  in Eq. 1.3-39. The steps described above are then repeated to find the perturbations in  $v_r$ ,  $v_z$  and  $p$ . The final result for  $F$  is

$$F = \frac{3\pi R^4 \mu(-\dot{h})}{8h^3} \left[ 1 + \frac{5\rho h(-\dot{h})}{7\mu} + \frac{2\rho h^2 \ddot{h}}{5\mu \dot{h}} \right] \quad (1.3-53)$$

This result<sup>7</sup> can be used to estimate the importance of inertial effects in lubrication squeeze films and in parallel plate plastometers.

### §1.4 SOLUTION OF ISOTHERMAL FLOW PROBLEMS BY USE OF THE STREAM FUNCTION

This section is devoted to the stream function as an important analytical tool for the solution of flow problems. For simplicity we restrict our attention to flows in which we may choose a coordinate system such that only two velocity components are nonzero. Following a brief introduction to the stream function we demonstrate its use in three illustrative examples in which we solve for the flow field around a translating sphere, a rising bubble, and a rotating sphere. The last example also introduces regular perturbation methods, which we will use in later chapters.

For illustrative purposes we start by considering two-dimensional "plane" flow, for which one may choose a rectangular coordinate system so that  $v_x = v_x(x, y, t)$ ,  $v_y = v_y(x, y, t)$ , and  $v_z = 0$ . The continuity equation then reads

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (1.4-1)$$

The form of this equation suggests the introduction of a function  $\psi(x, y, t)$  called the *stream function* with the property<sup>1</sup> that

$$v_x = - \frac{\partial \psi}{\partial y} \quad (1.4-2)$$

$$v_y = \frac{\partial \psi}{\partial x} \quad (1.4-3)$$

<sup>7</sup> S. Ishizawa, *JSME Bull.*, **9**, 533-550 (1966); D. C. Kuzma, *Appl. Sci. Res.*, **A18**, 15-20 (1967); A. F. Jones and S. D. R. Wilson, *J. Lubr. Technol.*, **97**, 101-104 (1975); R. J. Grimm, *Appl. Sci. Res.*, **32**, 149-166 (1976).

<sup>1</sup> Some authors introduce a stream function equal to the negative of the one introduced here. Our sign convention agrees with that of H. Lamb, *Hydrodynamics*, 6th ed., Cambridge University Press, Cambridge (1932); L. M. Milne-Thomson, *Theoretical Hydrodynamics* Macmillan, New York (1955); and R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York (1960), §4.2.

# CHAPTER 4

## THE GENERALIZED NEWTONIAN FLUID

This chapter is devoted to the generalized Newtonian fluid, which results from a minor modification of the Newtonian fluid constitutive equation given in Eq. 1.2-2. This equation incorporates the idea of a shear-rate-dependent viscosity, and hence can describe the non-Newtonian viscosity curves shown in Figs. 3.3-1 to 4. It cannot, however, describe normal stress effects or time-dependent elastic effects. There are many industrial flow problems in which the non-Newtonian viscosity effects are of paramount importance, and hence the generalized Newtonian fluid is useful; in addition it is a relatively simple constitutive equation, and many problems have been solved using it.

In §4.1 we introduce the model and give several useful empiricisms for the non-Newtonian viscosity, in particular, the “power-law model.” Then in §4.2 we show how to use the power-law model by going through a series of illustrative examples; the problems in these examples are sufficiently simple that analytical solutions may be obtained. For more complicated problems a variational principle is available, and it is discussed in §4.3. Up to this point it is presumed that the flow system is isothermal. Many industrial non-Newtonian problems are nonisothermal, and §4.4 provides a brief introduction to this rather large subject; particular attention is paid to viscous heating, because this effect is generally present in plastics processing operations and in lubrication problems. In §§4.2 to 4.4 all the illustrative examples are worked out for the power-law model; in §4.5 we discuss several other models and illustrate their use. Finally, in §4.6 we summarize the limitations on the generalized Newtonian fluid model and discuss the extent to which its use can be legitimized.

### **§4.1 THE GENERALIZED NEWTONIAN FLUID AND USEFUL EMPIRICISMS FOR THE NON-NEWTONIAN VISCOSITY**

For the industrial chemical engineer, the most important property of macromolecular fluids discussed in Chapter 3 is the non-Newtonian viscosity—that is, the fact that the viscosity of the fluid changes with the shear rate. Since for some fluids the viscosity can change by a factor of 10, 100, or even 1000, it is evident that such an enormous change cannot be ignored in pipe flow calculations, lubrication problems, design of on-line viscometers, extruder operation, and polymer processing calculations. Therefore it is not surprising that one of the earliest empiricisms to be introduced was a modification of

Newton's law of viscosity in which the viscosity is allowed to vary with the shear rate. That is, for the elementary flow  $v_x = v_x(y)$ ,  $v_y = 0$ ,  $v_z = 0$ , the early rheologists replaced

*Newtonian fluid:* 
$$\tau_{yx} = -\mu \frac{dv_x}{dy}, \quad \mu = \text{a constant for a given temperature, pressure, and composition} \quad (4.1-1)$$

by the empiricism:<sup>1</sup>

*Generalized Newtonian fluid:* 
$$\tau_{yx} = -\eta \frac{dv_x}{dy}, \quad \eta = \text{a function of } |dv_x/dy| \quad (4.1-2)$$

Absolute value signs have been used at the right of Eq. 4.1-2, since one would expect the change in viscosity to depend on the magnitude but not on the sign of the velocity gradient. Having written Eq. 4.1-2 one can then introduce various empiricisms to describe the experimental non-Newtonian viscosity curves.

We now wish to extend the ideas above to an arbitrary flow. First, for an incompressible Newtonian fluid, for any flow field  $\mathbf{v} = \mathbf{v}(x, y, z, t)$  we have:

*Incompressible Newtonian fluid:* 
$$\tau = -\mu \dot{\gamma} \quad \mu = \text{a constant for a given temperature, pressure, and composition} \quad (4.1-3)$$

in which  $\dot{\gamma}$  is the rate-of-strain tensor  $\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger$  (cf. Eq. 1.2-2). To include the idea of a non-Newtonian viscosity, we write<sup>1</sup>

*Incompressible generalized Newtonian fluid:* 
$$\tau = -\eta \dot{\gamma} \quad \eta = \text{a function of the scalar invariants of } \dot{\gamma} \quad (4.1-4)$$

If the non-Newtonian viscosity, a scalar, is to depend on the tensor  $\dot{\gamma}$ , then it must depend only on those particular combinations of components of the tensor that are not dependent on the coordinate system.<sup>2</sup> As described in §A.3 we may select as three independent combinations:

$$I = \sum_i \dot{\gamma}_{ii} \quad (4.1-5)$$

$$II = \sum_i \sum_j \dot{\gamma}_{ij} \dot{\gamma}_{ji} \quad (4.1-6)$$

$$III = \sum_i \sum_j \sum_k \dot{\gamma}_{ij} \dot{\gamma}_{jk} \dot{\gamma}_{ki} \quad (4.1-7)$$

For an incompressible fluid  $I = 2(\nabla \cdot \mathbf{v}) = 0$ . For shearing flows  $III$  turns out to be zero (see Problem 4B.8); since, as we will point out later, Eq. 4.1-4 should be used only for shearing

<sup>1</sup> M. Reiner, *Deformation, Strain, and Flow*, Interscience, New York (1960).

<sup>2</sup> K. Hohenemser and W. Prager, *J. Appl. Phys.*, **12**, 216-226 (1932); J. G. Oldroyd, *Proc. Camb. Phil. Soc.*, **45**, 595-611 (1949); **47**, 410-418 (1950).

flows, or at least flows that are very nearly shearing, omitting  $III$  from any further consideration is not a serious restriction. Hence  $\eta$  is taken to depend only on  $II$ . Actually we shall prefer to use  $\dot{\gamma}$ , the magnitude of the rate-of-strain tensor  $\dot{\gamma}$ , instead of  $II$

$$\dot{\gamma} = \sqrt{\frac{1}{2} \sum_i \sum_j \dot{\gamma}_{ij} \dot{\gamma}_{ji}} = \sqrt{\frac{1}{2} II} \tag{4.1-8}$$

In taking the square root one must affix the proper sign so that  $\dot{\gamma}$  will be positive. In shearing flows  $\dot{\gamma}$  is called the “shear rate.”

Equation 4.1-4 with  $\eta = \eta(\dot{\gamma})$  is then the starting point for all of the calculations in this chapter. Its principal usefulness is for calculating flow rates and shearing forces in steady-state shear flows, such as:

- a. Tube flow.
- b. Axial annular flow.
- c. Tangential annular flow.
- d. Helical annular flow.
- e. Flow between parallel planes.
- f. Flow between rotating disks.
- g. Cone-and-plate flow.

Although Eq. 4.1-4 gives correct results for flow rates and shearing forces in steady shear flows, we hasten to add that engineers have not hesitated to apply this equation to somewhat more complicated flows and systems slowly varying with time. A thorough assessment of the errors inherent in such calculations is not available, but one feels intuitively that such a practice probably represents good engineering empiricism. We give a few examples of this presently, and then in §4.6 we discuss the limits on the use of the model.

Let us now turn to the empiricisms for  $\eta(\dot{\gamma})$ . From Chapter 3 we know the appearance of the  $\eta$  vs.  $\dot{\gamma}$  curves. Although for some problems one can use the raw data for  $\eta(\dot{\gamma})$ , it is often useful to make calculations and derivations with simple empirical equations for  $\eta(\dot{\gamma})$  that are known to describe the experimental data with sufficient accuracy. Many such empiricisms are available, and we make no attempt at completeness. We cite only two here and mention some others in Table 4.5-1.

a. The Carreau–Yasuda Model (Parameters:  $\eta_0, \eta_\infty, \lambda, n, a$ )

This five-parameter model has sufficient flexibility to fit a wide variety of experimental  $\eta(\dot{\gamma})$  curves; it has proven to be useful for numerical calculations in which one needs an analytical expression for the non-Newtonian viscosity curve. The model is<sup>3,4</sup>

$$\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = [1 + (\lambda \dot{\gamma})^a]^{(n-1)/a} \tag{4.1-9}$$

Here  $\eta_0$  is the zero-shear-rate viscosity,  $\eta_\infty$  is the infinite-shear-rate viscosity,  $\lambda$  is a time constant,  $n$  is the “power-law exponent” (since it describes the slope of  $(\eta - \eta_\infty)/(\eta_0 - \eta_\infty)$  in the “power-law region”), and  $a$  is a dimensionless parameter that describes the transition region between the zero-shear-rate region and the power-law region. Some examples of

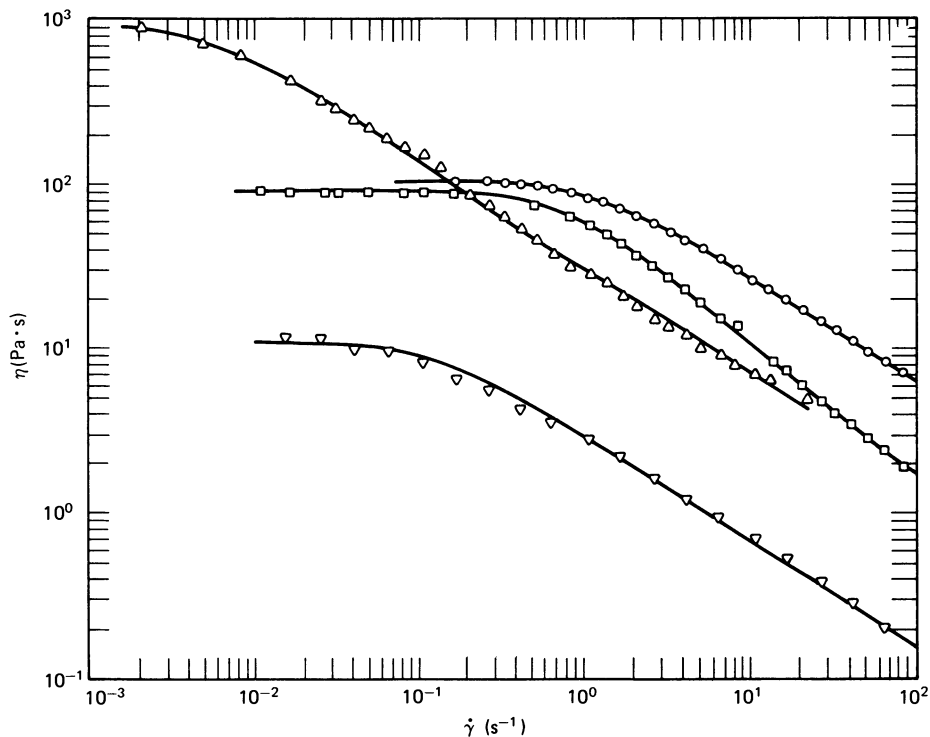


FIGURE 4.1-1. Non-Newtonian viscosity of three polymer solutions and a soap solution as fitted by the Carreau viscosity equation (Eq. 4.1-9, with  $a = 2$ ). [R. B. Bird, O. Hassager, and S. I. Abdel-Khalik, *AIChE J.*, **20**, 1041-1066 (1974).]  $\Delta$  2% polyisobutylene in Primol 355; data of J. D. Huppler, E. Ashare, and L. A. Holmes, *Trans. Soc. Rheol.*, **11**, 159-179 (1968):  $\eta_0 = 9.23 \times 10^2$  Pa·s,  $\eta_\infty = 1.50 \times 10^{-1}$  Pa·s,  $\lambda = 191$  s,  $n = 0.358$ .  $\circ$  5% polystyrene in Aroclor 1242; data of E. Ashare, Ph.D. Thesis, University of Wisconsin, Madison (1968):  $\eta_0 = 1.01 \times 10^2$  Pa·s,  $\eta_\infty = 5.9 \times 10^{-2}$  Pa·s,  $\lambda = 0.84$  s,  $n = 0.364$ .  $\nabla$  0.75% polyacrylamide (Separan-30) in a 95/5 mixture by weight of water and glycerin; data of B. D. Marsh (1967), as cited by P. J. Carreau, I. F. Macdonald, and R. B. Bird, *Chem. Eng. Sci.*, **23**, 901-911 (1968):  $\eta_0 = 10.6$  Pa·s,  $\eta_\infty = 10^{-2}$  Pa·s,  $\lambda = 8.04$  s,  $n = 0.364$ .  $\square$  7% aluminum soap in decalin and *m*-cresol; data of J. D. Huppler, E. Ashare, and L. A. Holmes, *loc. cit.*:  $\eta_0 = 89.6$  Pa·s,  $\eta_\infty = 10^{-2}$  Pa·s,  $\lambda = 1.41$  s,  $n = 0.200$ .

curve-fitting of experimental data are given in Figs. 4.1-1, 2, and 3, and sample values of the parameters in the Carreau-Yasuda model are given in Table 4.1-1 for polystyrene solutions. For many concentrated polymer solutions and melts, good fits can be obtained for  $a = 2$  and  $\eta_\infty = 0$ ; then only three parameters  $\eta_0$ ,  $\lambda$ , and  $n$  need to be determined. Equation 4.1-9, with  $a = 2$ , is usually referred to as the Carreau equation,<sup>3</sup> since the parameter  $a$  was added later by Yasuda.<sup>4</sup>

#### b. The "Power-Law" Model<sup>5</sup> of Ostwald and de Waele (Parameters: $m$ and $n$ )

In almost all industrial problems the descending linear region (the "power-law region") of the plot of  $\log \eta$  vs.  $\log \dot{\gamma}$ , seen in Figs. 4.1-1, 2, and 3, is the most important

<sup>3</sup> P. J. Carreau, Ph.D. Thesis, University of Wisconsin, Madison (1968).

<sup>4</sup> K. Yasuda, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge (1979); K. Yasuda, R. C. Armstrong, and R. E. Cohen, *Rheol. Acta*, **20**, 163-178 (1981).

<sup>5</sup> W. Ostwald, *Kolloid-Z.*, **36**, 99-117 (1925); A. de Waele, *Oil Color Chem. Assoc. J.* **6**, 33-88 (1923).

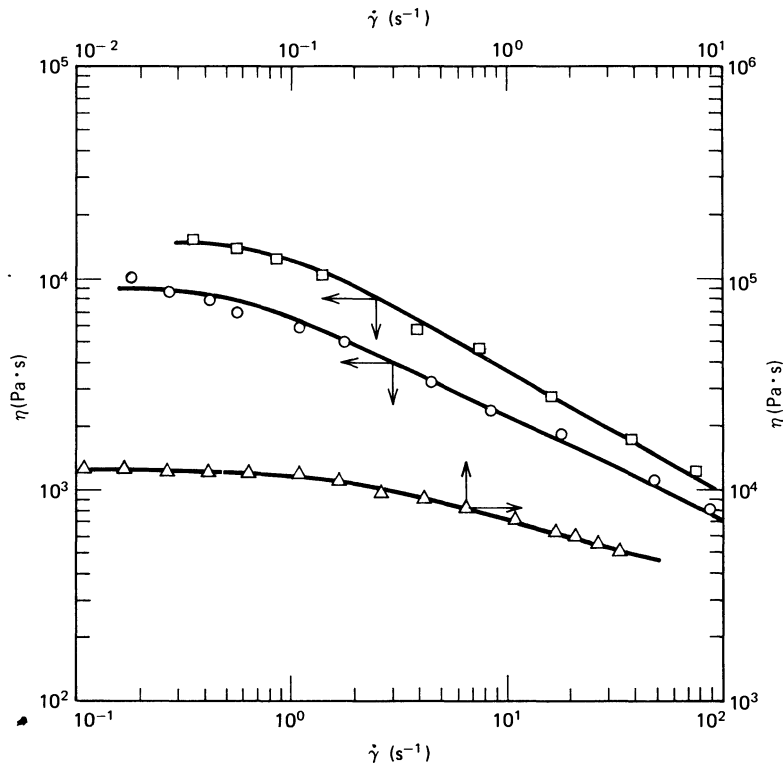


FIGURE 4.1-2. Non-Newtonian viscosity of three polymer melts as described by the Carreau viscosity equation (Eq. 4.1-9, with  $a = 2$ ). [S. I. Abdel-Khalik, O. Hassager, and R. B. Bird, *Polym. Eng. Sci.*, **14**, 859–867 (1974).]  $\square$  Polystyrene at 453 K; data of T. F. Ballenger, I.-J. Chen, J. W. Crowder, G. E. Hagler, D. C. Bogue, and J. L. White, *Trans. Soc. Rheol.*, **15**, 195–215 (1971):  $\eta_0 = 1.48 \times 10^4 \text{ Pa}\cdot\text{s}$ ,  $\eta_\infty = 0$ ,  $\lambda = 1.04 \text{ s}$ ,  $n = 0.938$ .  $\circ$  High-density polyethylene at 443 K; data of Ballenger, *et al.*, *loc. cit.*:  $\eta_0 = 8.92 \times 10^3 \text{ Pa}\cdot\text{s}$ ,  $\eta_\infty = 0$ ,  $\lambda = 1.58 \text{ s}$ ,  $n = 0.496$ .  $\triangle$  Phenoxy-A at 485 K; data of B. D. Marsh as cited by P. J. Carreau, I. F. Macdonald, and R. B. Bird, *Chem. Eng. Sci.*, **23**, 901–911 (1968):  $\eta_0 = 1.24 \times 10^4 \text{ Pa}\cdot\text{s}$ ,  $\eta_\infty = 0$ ,  $\lambda = 7.44 \text{ s}$ ,  $n = 0.728$ .

region. In fact, for many inexpensive viscometers and for many fluids, it is almost impossible to obtain data for the horizontal region of the  $\eta(\dot{\gamma})$  curve. The tilted straight line can be described by a simple “power-law” expression:

$$\eta = m\dot{\gamma}^{n-1} \quad (4.1-10)$$

which contains two parameters:  $m$  (with units of  $\text{Pa}\cdot\text{s}^n$ ), and  $n$  (dimensionless). Equation 4.1-10 may also be regarded as the limiting expression for high shear rates obtained from Eq. 4.1-9 (with  $\eta_\infty = 0$ ); it is then evident that the exponent  $n$  in the power-law model has the same meaning as the  $n$  in the Carreau–Yasuda equation, and that the  $m$  of the power law (sometimes referred to as the “consistency index”) is  $\eta_0 \lambda^{n-1}$ . When  $n = 1$  and  $m = \mu$  the Newtonian fluid is recovered. If  $n < 1$ , the fluid is said to be “pseudoplastic” or “shear thinning,” and if  $n > 1$ , the fluid is called “dilatant” or “shear thickening.”<sup>6</sup> Some sample

<sup>6</sup> M. Reiner, *op cit.*, pp. 306–308, and others use the term “dilatancy” to describe the change in volume of granular masses necessitated by a distortion. The standard example is the apparent drying of wet sand when stepped on.

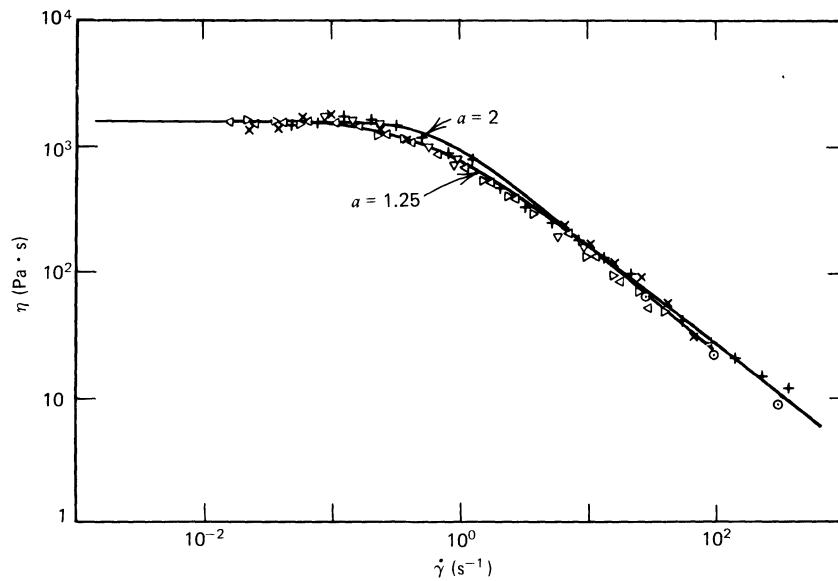


FIGURE 4.1-3. Non-Newtonian viscosity of a solution of linear, monodisperse polystyrene in 1-chloronaphthalene as fitted by the Carreau-Yasuda viscosity equation in Eq. 4.1-9. The concentration of polymer is 0.15 g/ml, and  $\bar{M}_w = 2 \times 10^6$ . Fits are shown for two different choices of the parameter  $a$ . For  $a = 1.25$ , the other model parameters are  $\eta_0 = 1400 \text{ Pa} \cdot \text{s}$ ,  $\lambda = 1.60 \text{ s}$ ,  $n = 0.2$ , and  $\eta_\infty = 0$ . [K. Yasuda, Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge (1979).]

TABLE 4.1-1

Parameters in Carreau-Yasuda Model for Some Solutions of Linear Polystyrene in 1-Chloronaphthalene<sup>a</sup>

Properties of Solution		Parameters in Eq. 4.1-9 ( $\eta_\infty$ is taken to be zero)			
$\bar{M}_w$ (g/mol)	$c$ (g/ml)	$\eta_0$ (Pa · s)	$\lambda$ (s)	$n$ (---)	$a$ (---)
$2 \times 10^6$	0.15	1400	1.60	0.200	1.25
$2 \times 10^6$	0.088	90	$3.79 \times 10^{-1}$	0.265	0.98
$3.9 \times 10^5$	0.45	8080	1.109	0.304	2
$3.9 \times 10^5$	0.30	135	$3.61 \times 10^{-2}$	0.305	2
$1.1 \times 10^5$	0.52	1180	$9.24 \times 10^{-2}$	0.441	2
$1.1 \times 10^5$	0.45	166	$1.73 \times 10^{-2}$	0.538	2
$3.7 \times 10^4$	0.62	3930	$1 \times 10^{-1}$	0.217	2

<sup>a</sup> Values of the parameters are taken from K. Yasuda, R. C. Armstrong, and R. E. Cohen, *Rheol. Acta*, **20**, 163-178 (1981).

**TABLE 4.1-2**  
**Power-Law Parameters for Aqueous Solutions<sup>a</sup>**

Solution	Temperature (K)	$m$ (Pa · s <sup><i>n</i></sup> )	$n$ (—)
0.5% Hydroxyethylcellulose	293	0.84	0.509
	313	0.30	0.595
	333	0.136	0.645
2.0% Hydroxyethylcellulose	293	93.5	0.189
	313	59.7	0.223
	333	38.5	0.254
1.0% Polyethylene oxide	293	0.994	0.532
	313	0.706	0.544
	333	0.486	0.599

<sup>a</sup> R. M. Turian, Ph.D. Thesis, University of Wisconsin, Madison (1964), pp. 142–148.

numerical values of  $m$  and  $n$  are given in Table 4.1-2; note that  $m$  and  $n$  are both temperature dependent, the parameter  $m$  decreasing rapidly with increasing temperature.

The power-law model for  $\eta(\dot{\gamma})$  is the most well-known and widely-used empiricism in engineering work, because a wide variety of flow problems have been solved analytically for it. One can often get a rough estimate of the effect of the non-Newtonian viscosity by making a calculation based on the power-law model. However its shortcomings must not be overlooked: (i) it cannot describe the viscosity for very small shear rates, and in some problems this can lead to large errors, (ii) a characteristic time and a characteristic viscosity cannot be constructed from the parameters  $m$  and  $n$  alone, and this can be awkward in pursuing dimensional-analysis arguments (see §2.8), and (iii) there is no way to relate the parameters  $m$  and  $n$  to molecular weight and concentration, since the standard correlations are in terms of  $\eta_0$  and  $\eta^*(\omega)$  (see §3.6). Keep in mind that Eqs. 4.1-9 and 10 are just empirical curve fits of the experimental  $\eta(\dot{\gamma})$  curves. Because of the widespread use of the power-law model we illustrate its application in the next section.

## §4.2 ISOTHERMAL FLOW PROBLEMS

The procedure for the solution of elementary flow problems for generalized Newtonian fluids is exactly the same as for Newtonian fluids, except that the mathematics is more awkward because of the additional complexity introduced by the non-Newtonian viscosity function. In this section we give a series of examples to illustrate the solution procedure for power-law fluids. A summary of power-law solutions is given in Table 4.2-1.

**EXAMPLE 4.2-1** Flow of a Power-Law Fluid in a Straight Circular Tube and in a Slightly Tapered Tube

Rework Examples 1.3-2 and 1.3-3 for the power-law model.

**SOLUTION** For a straight tube of uniform cross section we postulate a solution of the form  $v_z = v_z(r)$ ,  $v_\theta = 0$ ,  $v_r = 0$ , and  $\mathcal{P} = \mathcal{P}(z)$ . The  $z$ -component of the equation of motion, in terms of  $\tau$ , is

$$0 = -\frac{d\mathcal{P}}{dz} - \frac{1}{r} \frac{d}{dr} (r\tau_{rz}) \tag{4.2-1}$$

**TABLE 4.2-1**  
**Solutions to Flow Problems for Power-Law Model**

Problem	Solution
Axial flow through a slit of width $W$ , thickness $2B$ , and length $L$ under a pressure drop $\mathcal{P}_0 - \mathcal{P}_L$ (with $B \ll W \ll L$ )	$Q = \frac{2WB^2}{(1/n) + 2} \left( \frac{(\mathcal{P}_0 - \mathcal{P}_L)B}{mL} \right)^{1/n}$ (A)
Axial flow through a circular tube of radius $R$ and length $L$ under a pressure drop $\mathcal{P}_0 - \mathcal{P}_L$ (with $R \ll L$ )	$Q = \frac{\pi R^3}{(1/n) + 3} \left( \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2mL} \right)^{1/n}$ (B)
Axial flow through an annulus with inner and outer radii $\kappa R$ and $R$ , and length $L$ , under a pressure drop $\mathcal{P}_0 - \mathcal{P}_L$ ( $r = \beta R$ is the location of the maximum in the velocity profile—see Table 4.2-2)	$Q = \frac{\pi R^3}{(1/n) + 3} \left( \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{2mL} \right)^{1/n} \cdot [(1 - \beta^2)^{1+(1/n)} - \kappa^{1-(1/n)}(\beta^2 - \kappa^2)^{1+(1/n)}]$ (C)
Axial flow through an annulus with no axial pressure drop; inner cylinder moves with velocity $V$ ( $\kappa, R, L$ have same meanings as in Eq. C)	$Q = \pi R^2 V \frac{(3 - (1/n))(1 - \kappa^2) - 2(1 - \kappa^{3-(1/n)})}{(3 - (1/n))(1 - \kappa^{1-(1/n)})}$ (D)
Applied torque $\mathcal{T}$ in tangential annular flow between two coaxial cylinders the inner one of which is being made to rotate at an angular velocity $W$ ( $\kappa, R, L$ have the same meanings as in Eq. C)	$\mathcal{T} = 2\pi(\kappa R)^2 mL \left( \frac{2W/n}{1 - \kappa^{2/n}} \right)^n$ (E)
Radial flow between two parallel disks separated by $2B$ , with pressure drop $\mathcal{P}_1 - \mathcal{P}_2$ over the distance from $r = R_1$ to $r = R_2$	$Q = \frac{4\pi B^2}{(1/n) + 2} \left( \frac{(\mathcal{P}_1 - \mathcal{P}_2)B(1 - n)}{m(R_2^{1-n} - R_1^{1-n})} \right)^{1/n}$ (F) <sup>a</sup>
Squeezing flow between two circular disks of radius $R$ , with applied force $F(t)$ and instantaneous disk separation $2h$ (the instantaneous plate velocity is $\dot{h}$ )	$F(t) = \frac{(-\dot{h})^n}{h^{2n+1}} \left( \frac{2n+1}{2n} \right)^n \left( \frac{\pi m R^{n+3}}{n+3} \right)$ (G) <sup>b</sup>

<sup>a</sup> Lubrication-approximation expression obtained by applying Eq. A locally in the space between disks.

<sup>b</sup> Approximate expression obtained by applying Eq. A locally and using a quasi-steady-state assumption.

or using the arguments given in Example 1.3-2

$$\frac{1}{r} \frac{d}{dr} (r\tau_{rz}) = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} \quad (4.2-2)$$

This may be integrated to give:

$$\tau_{rz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)r}{2L} + \frac{C_1}{r} \quad (4.2-3)$$

The constant  $C_1$  has to be zero since one does not expect to have an infinite stress at the tube axis. Equation 4.2-3 then is the result of the application of the equation of motion (i.e., conservation of momentum). Equation 4.2-3 may be written in the alternative form

$$\tau_{rz} = \tau_R \cdot \frac{r}{R} \quad (4.2-4)$$

where  $\tau_R$  is the wall shear stress; that is  $\tau_{rz} = \tau_R$  at  $r = R$ .

Next we have to use the power-law equation for the stress, as given by Eqs. 4.1-4 and 4.1-10. In the latter equation  $\dot{\gamma}$ , which must be a positive quantity, is given by  $(-dv_z/dr)$ . Then

$$\tau_{rz} = -\eta \frac{dv_z}{dr} = -m\dot{\gamma}^{n-1} \frac{dv_z}{dr} = m \left( -\frac{dv_z}{dr} \right)^n \quad (4.2-5)$$

Combination of Eqs. 4.2-4 and 4.2-5 then gives the differential equation for  $v_z$ :

$$m \left( -\frac{dv_z}{dr} \right)^n = \tau_R \cdot \frac{r}{R} \quad (4.2-6)$$

Taking the  $n$ th root of both sides and integrating the first-order, separable equation gives

$$v_z = -\left( \frac{\tau_R}{mR} \right)^{1/n} \frac{r^{(1/n)+1}}{(1/n)+1} + C_2 \quad (4.2-7)$$

The constant  $C_2$  is evaluated by requiring that  $v_z$  be zero at  $r = R$ . One then gets finally

$$v_z = \left( \frac{\tau_R}{m} \right)^{1/n} \frac{R}{(1/n)+1} \left[ 1 - \left( \frac{r}{R} \right)^{(1/n)+1} \right] \quad (4.2-8)$$

For  $n < 1$  this gives a velocity profile that is flatter than the parabolic profile in Eq. 1.3-10 for Newtonian fluids (see Fig. 4.2-1).

It is then easy to get the volume rate of flow  $Q$ :

$$\begin{aligned} Q &= \int_0^{2\pi} \int_0^R v_z r \, dr \, d\theta \\ &= 2\pi R^2 \int_0^1 v_z \cdot \frac{r}{R} \, d\left(\frac{r}{R}\right) \\ &= \frac{\pi R^3}{(1/n)+3} \left( \frac{\tau_R}{m} \right)^{1/n} \\ &= \frac{\pi R^3}{(1/n)+3} \left[ \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2mL} \right]^{1/n} \end{aligned} \quad (4.2-9)$$

For  $n = 1$  and  $m = \mu$  this reduces to the Hagen-Poiseuille equation for Newtonian fluids.<sup>1</sup>

<sup>1</sup> Laminar-turbulent transition has been studied by D. W. Dodge and A. B. Metzner, *AIChE J.*, **5**, 189-204 (1959); they found that the laminar-turbulent transition occurred in the modified Reynolds number range  $2100 < \text{Re}_n < 3100$ , where  $\text{Re}_n = (D^n \langle v_z \rangle^{2-n} \rho / m)(3 + (1/n))^{-n} 2^{3-n}$  for power-law fluids. For other studies see N. W. Ryan and M. M. Johnson, *AIChE J.*, **5**, 433-435 (1959); R. W. Hanks, *ibid.*, **9**, 306-309 (1963); D. M. Meter, *ibid.*, **10**, 881-884 (1964).

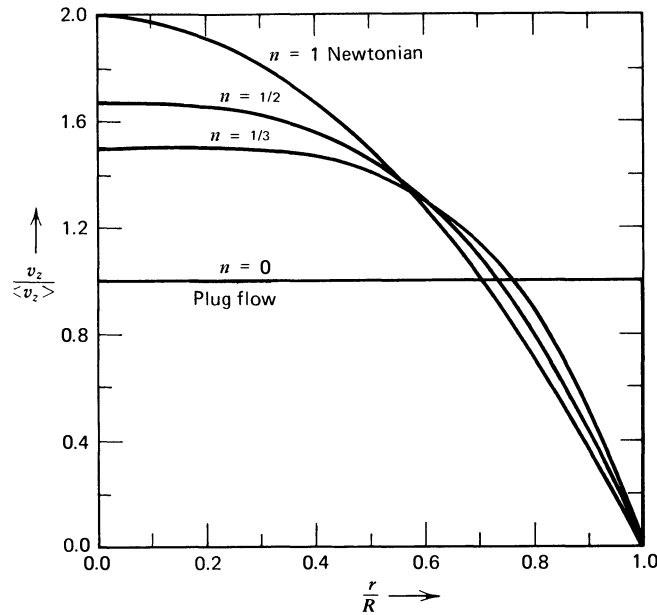


FIGURE 4.2-1. Tube flow velocity profiles for a power-law fluid from Eq. 4.2-8. Note that the profiles become increasingly flatter as  $n$  decreases;  $n = 0$  corresponds to plug flow. The Newtonian (parabolic) profile is shown as  $n = 1$ .

The shear rate at the tube wall,  $\dot{\gamma}_R = (-dv_z/dr)|_{r=R}$  is a quantity that is sometimes of interest in viscometry. An expression for this quantity can be obtained from Eqs. 4.2-6 and 9; it is found that  $\dot{\gamma}_R = [(3n + 1)/4n]\dot{\gamma}_a$ , where  $\dot{\gamma}_a = 4\langle v_z \rangle/R$  is the "apparent shear rate" (note that for a Newtonian fluid  $\dot{\gamma}_a$  is identical to  $\dot{\gamma}_R$ ).

As already pointed out, the power-law model gives an unreasonably high value for the viscosity  $\eta(\dot{\gamma})$  for small values of the shear rate. It is therefore appropriate to ask what limitations have to be placed on Eq. 4.2-9 because of this defect of the power law. It is shown in Problem 4B.13 that the power-law expression for  $Q$  can be expected to be reliable when  $\tau_R \gg \eta_0 \dot{\gamma}_0$ , where  $\eta_0$  is the zero-shear-rate viscosity and  $\dot{\gamma}_0$  is the value of the shear rate at which shear thinning begins (see §3.6).

For a tapered tube whose radius decreases linearly from  $R_0$  at  $z = 0$  to  $R_L$  at  $z = L$ , we get by applying Eq. 4.2-9 locally for a small segment  $dz$  of the tube

$$\mathcal{P}_0 - \mathcal{P}_L = \frac{2mL}{3n} \left[ \frac{Q}{\pi} \left( \frac{1}{n} + 3 \right) \right]^n \left( \frac{R_L^{-3n} - R_0^{-3n}}{R_0 - R_L} \right) \quad (4.2-10)$$

This gives the relation between pressure drop and volume rate of flow<sup>2</sup>.

Sutterby<sup>3</sup> found that the use of a generalized Newtonian fluid model described adequately the  $Q$  vs.  $\mathcal{P}_0 - \mathcal{P}_L$  relationship for the slow flow of polymer solutions in a converging tube. For very fast flow, on the other hand, the data were well described by the results of an ideal (inviscid) fluid calculation; this is perhaps not too surprising since the inviscid fluid corresponds to  $Re \rightarrow \infty$ .

The flow of non-Newtonian fluids in tapered tubes has also been studied by Oka and Murata.<sup>4</sup>

<sup>2</sup> J. M. McKelvey, V. Maire, and F. Haupt, *Chem. Eng.*, **83**, 94-102 (1976).

<sup>3</sup> J. L. Sutterby, *Trans. Soc. Rheol.*, **9**:2, 227-241 (1965).

<sup>4</sup> S. Oka, *Zairyō*, **14**, 241-244 (1965); S. Oka and T. Murata, *Jpn. J. Appl. Phys.*, **8**, 5-8 (1969); S. Oka, *Biorheology*, **10**, 207-212 (1973).

**EXAMPLE 4.2-2** Thickness of a Film of Polymer Solution Flowing Down an Inclined Plate

Obtain an expression for the thickness of a film of polymer solution as it flows down an inclined plate making an angle  $\alpha$  with the vertical. Use the power-law fluid model for the derivation. Take the origin of coordinates to be such that  $x = 0$  at the film surface and  $x = \delta$  at the plate; the film extends along the plate from  $z = 0$  to  $z = L$ .

**SOLUTION** From the equation of motion we get

$$0 = -\frac{d\tau_{xz}}{dx} + \rho g \cos \alpha \quad (4.2-11)$$

When this is integrated, using the boundary condition that  $\tau_{xz} = 0$  at the liquid-air interface at  $x = 0$ , we get

$$\tau_{xz} = \rho g x \cos \alpha \quad (4.2-12)$$

We know that  $v_z$  decreases with increasing  $x$ , so that  $\dot{\gamma}$  will be taken to be  $(-dv_z/dx)$  in order to ensure that  $\dot{\gamma}$  be a positive quantity. Then the power law gives the following expression for  $\tau_{xz}$ :

$$\begin{aligned} \tau_{xz} &= -\eta \frac{dv_z}{dx} = -m \left( -\frac{dv_z}{dx} \right)^{n-1} \frac{dv_z}{dx} \\ &= +m \left( -\frac{dv_z}{dx} \right)^n \end{aligned} \quad (4.2-13)$$

Combination of Eqs. 4.2-12 and 4.2-13 then gives

$$-\frac{dv_z}{dx} = \left( \frac{\rho g x \cos \alpha}{m} \right)^{1/n} \quad (4.2-14)$$

Integration with the boundary condition that  $v_z = 0$  at  $x = \delta$  gives

$$v_z = \left( \frac{\rho g \delta \cos \alpha}{m} \right)^{1/n} \frac{\delta}{(1/n) + 1} \left[ 1 - \left( \frac{x}{\delta} \right)^{(1/n)+1} \right] \quad (4.2-15)$$

for the velocity distribution.

Integration over the cross section of flow (thickness  $\delta$  and width  $W$ ) gives for the volume rate of flow

$$Q = \frac{W\delta^2}{(1/n) + 2} \left( \frac{\rho g \delta \cos \alpha}{m} \right)^{1/n} \quad (4.2-16)$$

Solution for the film thickness then gives

$$\delta = \left( \frac{m}{\rho g \cos \alpha} \right)^{1/(2n+1)} \left( \frac{Q[(1/n) + 2]}{W} \right)^{n/(2n+1)} \quad (4.2-17)$$

This shows how the film thickness depends on the flow rate  $Q$ , the fluid properties  $m$  and  $n$ , and the angle of inclination  $\alpha$ .

**EXAMPLE 4.2-3** Plane Couette Flow<sup>5</sup>

A macromolecular fluid is confined to the space between two horizontal planes ( $x = 0$  and  $x = B$ ) the upper one of which is moving in the positive  $z$ -direction with a constant speed  $V$ . In addition there is a pressure gradient in the  $z$ -direction, the pressure at  $z = 0$  being  $p_0$  and that at  $z = L$  being  $p_L$ . Obtain an expression for the volume rate of flow in the  $z$ -direction in the slit that results from the combined effects of the motion of the upper plate and the pressure gradient. Ignore the effect of gravity.

Problems of this type arise in diverse processing operations, such as in certain types of extruders, and in various lubrication problems.

**SOLUTION** For this flow the equation of motion is

$$0 = -\frac{dp}{dz} - \frac{d\tau_{xz}}{dx} \quad (4.2-18)$$

which may be integrated to give

$$\tau_{xz} = -\left[\frac{(p_0 - p_L)B}{L}\right](\beta - \xi) \quad (4.2-19)$$

where  $\beta$  is a constant of integration, and  $\xi = x/B$ .

Equation 4.1-4 gives

$$\tau_{xz} = -\eta \frac{dv_z}{dx} \quad (4.2-20)$$

By way of illustration, we use the power law for  $\eta$ . Two cases have to be distinguished here:

*Case I:* There is no maximum in the velocity profile  $v_z(x)$

*Case II:* There is a maximum in the velocity profile  $v_z(x)$

We consider only Case I, although the final results for Case II will be given.

For Case I,  $\eta = m(dv_z/dx)^{n-1}$  and hence Eq. 4.2-20 becomes

$$\tau_{xz} = -m\left(\frac{dv_z}{dx}\right)^n \quad (4.2-21)$$

Combination of Eqs. 4.2-19 and 4.2-21 gives an equation for  $v_z$  as a function of  $x$ . Integration of this equation and application of the boundary conditions  $v_z = 0$  at  $x = 0$  and  $v_z = V$  at  $x = B$  gives

$$\phi(\xi) = \frac{v_z}{V} = \frac{\beta^{s+1} - (\beta - \xi)^{s+1}}{\beta^{s+1} - (\beta - 1)^{s+1}} \quad (4.2-22)$$

where  $\xi = x/B$ ,  $s = 1/n$ , and  $\beta$  is a dimensionless parameter given by

$$\begin{aligned} \Lambda &\equiv \frac{(p_0 - p_L)B}{mL} \left(\frac{B}{V}\right)^{1/s} \\ &= \left[\frac{s+1}{\beta^{s+1} - (\beta-1)^{s+1}}\right]^{1/s} \quad \text{Case I: } \Lambda \leq (s+1)^{1/s} \end{aligned} \quad (4.2-23)$$

<sup>5</sup> R. W. Flumerfelt, M. W. Pierick, S. L. Cooper, and R. B. Bird, *Ind. Eng. Chem. Fundam.*, **8**, 354-357 (1969); earlier work was done on this problem by F. W. Kroesser and S. Middleman, *Polym. Eng. Sci.*, **5**, 230-234 (1965) and Z. Tadmor, *Polym. Eng. Sci.*, **6**, 203-212 (1966). Annular Couette flow has been studied by S. H. Lin and C. C. Hsu, *Ind. Eng. Chem. Fundam.*, **19**, 421-424 (1980).

Hence, knowing  $\beta = \beta(\Lambda, s)$  from Eq. 4.2-23, one can obtain the velocity profile from Eq. 4.2-22. The volume rate of flow between two planes of width  $W$  is then found to be

$$\begin{aligned} \frac{Q}{WBV} &= \int_0^1 \phi \, d\xi = \phi \xi \Big|_0^1 - \int_0^1 \frac{d\phi}{d\xi} \xi \, d\xi \\ &= -(\beta - 1) + \left( \frac{s+1}{s+2} \right) \frac{\beta^{s+2} - (\beta-1)^{s+2}}{\beta^{s+1} - (\beta-1)^{s+1}} \quad \text{Case I: } \Lambda \leq (s+1)^{1/s} \end{aligned} \quad (4.2-24)$$

with  $\beta$  determined from Eq. 4.2-23.

The results analogous to Eqs. 4.2-23 and 4.2-24 for Case II can be shown to be

$$\Lambda = \left[ \frac{s+1}{\beta^{s+1} - (1-\beta)^{s+1}} \right]^{1/s} \quad \text{Case II: } \Lambda \geq (s+1)^{1/s} \quad (4.2-25)$$

$$\frac{Q}{WBV} = (1-\beta) + \left( \frac{s+1}{s+2} \right) \frac{\beta^{s+2} + (1-\beta)^{s+2}}{\beta^{s+1} - (1-\beta)^{s+1}} \quad \text{Case II: } \Lambda \geq (s+1)^{1/s} \quad (4.2-26)$$

Hence, the choice of Case I or Case II formulas depends on whether  $\Lambda$  is larger or smaller than  $(s+1)^{1/s}$ . A table of  $\beta = \beta(\Lambda, s)$  has been prepared by Flumerfelt *et al.*<sup>5</sup> The dimensionless flow rate  $\Omega = Q/WBV$  is shown in Fig. 4.2-2; this chart is so constructed that it includes the case  $p_0 < p_L$  as well as  $p_0 > p_L$ .

In Case II there is a maximum or a minimum in the velocity profile. Near the maximum or minimum in the velocity the power law will overestimate the viscosity by a large amount. Errors in the volume flow rate  $Q$  may be large unless the bigger of  $|\tau_{xz}(x=0)|$  and  $|\tau_{xz}(x=B)|$  is much greater than  $\eta_0 \dot{\gamma}_0$ , where  $\dot{\gamma}_0$  is that shear rate at which shear thinning of the viscosity begins.

#### EXAMPLE 4.2-4 Axial Annular Flow<sup>6-8</sup>

Obtain the relation between the pressure drop and volume rate of flow for the pressure-driven flow of a power-law fluid through the annular gap between two coaxial cylinders of radii  $\kappa R$  and  $R$  (with  $\kappa < 1$ ). Let the maximum in the velocity distribution be located at  $r = \beta R$ , where  $\beta$  is a constant that has to be determined later.

**SOLUTION** We postulate that  $v_z = v_z(r)$ ,  $v_r = 0$ ,  $v_\theta = 0$ , and  $\mathcal{P} = \mathcal{P}(z)$ . Then the differential equation for  $\tau_{rz}$ , obtained from the  $z$ -component of the equation of motion, is found to be Eq. 4.2-1; this can be integrated to give Eq. 4.2-3, but the constant  $C_1$  cannot now be set equal to zero, because for this problem  $\kappa R \leq r \leq R$ . However we can rewrite Eq. 4.2-3 so that the constant  $\beta$  appears rather than  $C_1$ :

$$\tau_{rz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2L} \left( \frac{r}{R} - \beta^2 \frac{R}{r} \right) \quad (4.2-27)$$

<sup>6</sup> A. G. Fredrickson and R. B. Bird, *Ind. Eng. Chem.*, **50**, 347-352 (1958); erratum, *Ind. Eng. Chem. Fundam.*, **3**, 383 (1964).

<sup>7</sup> R. W. Hanks and K. M. Larsen, *Ind. Eng. Chem. Fundam.*, **18**, 33-35 (1979).

<sup>8</sup> For flow in converging annular regions see J. Parnaby and R. A. Worth, *Proc. Inst. Mech. Eng.*, **188**, 357-364 (1974), and J. F. Dijksman and E. P. W. Savenije, *Rheol. Acta*, **24**, 105-118 (1985); in the latter a special toroidal coordinate system is developed and used.

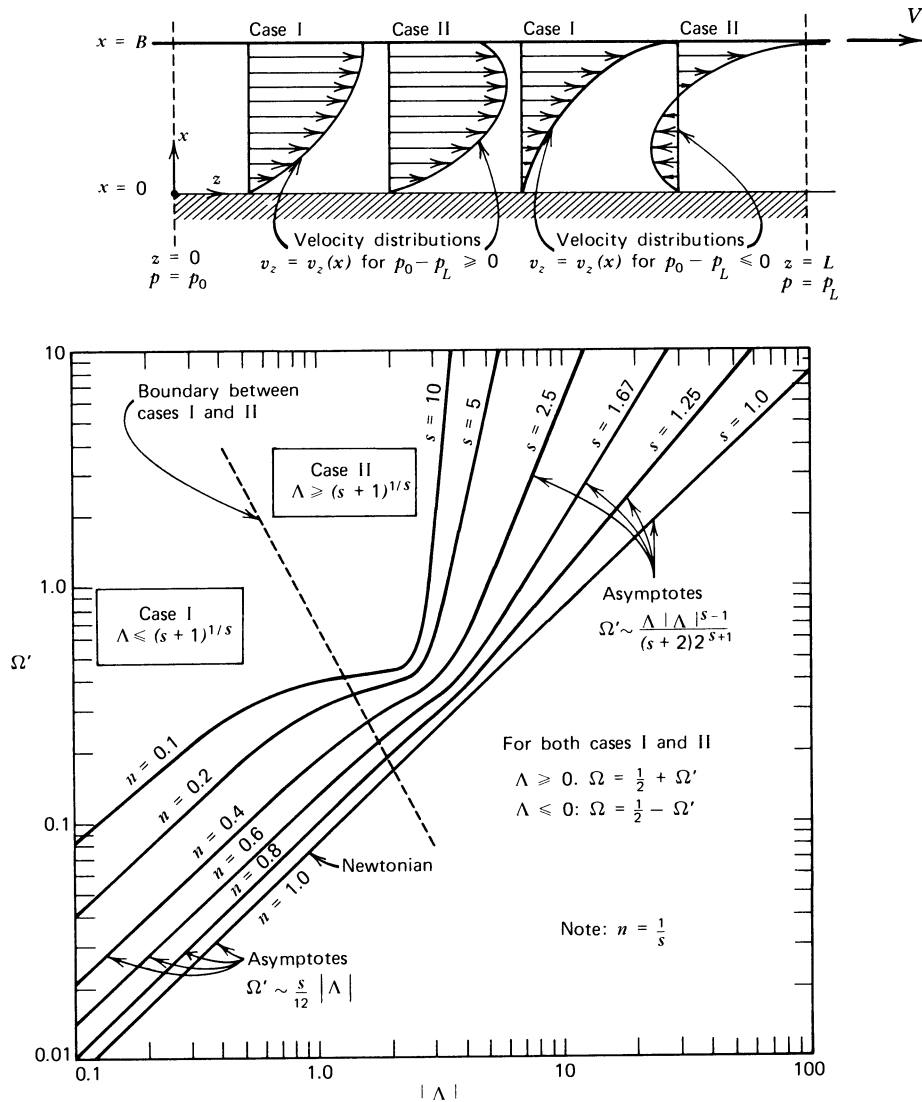


FIGURE 4.2-2. Dimensionless flow rate  $\Omega = Q/WBV$  as a function of  $\Lambda$  and  $n$  (or  $1/s$ ) for generalized Couette flow. [Reprinted with permission from R. W. Flumerfelt, M. W. Pierick, S. L. Cooper, and R. B. Bird, *Ind. Eng. Chem. Fundam.*, **8**, 354-387 (1969). Copyright by the American Chemical Society.]

The parameter  $\beta$  is then regarded as the “constant of integration.” The power-law expression for the shear stress is given by

$$\tau_{rz} = -m \left( \frac{dv_z}{dr} \right)^n, \quad \kappa R \leq r \leq \beta R \tag{4.2-28}$$

$$\tau_{rz} = m \left( -\frac{dv_z}{dr} \right)^n, \quad \beta R \leq r \leq R \tag{4.2-29}$$

Substitution of these expressions into Eq. 4.2-27 leads then to differential equations for the velocity distribution in the two regions. These equations may now be integrated and the boundary conditions used:  $v_z = 0$  at  $r = \kappa R$  and at  $r = R$ . This leads to

$$v_z = R \left[ \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2mL} \right]^s \int_{\kappa}^{\xi} \left( \frac{\beta^2}{\xi'} - \xi' \right)^s d\xi', \quad \kappa \leq \xi \leq \beta \tag{4.2-30}$$

$$v_z = R \left[ \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2mL} \right]^s \int_{\xi}^1 \left( \xi' - \frac{\beta^2}{\xi'} \right)^s d\xi', \quad \beta \leq \xi \leq 1 \tag{4.2-31}$$

in which  $\xi = r/R$  and  $s = 1/n$ .

Next the constant  $\beta$  is determined by requiring that Eqs. 4.2-30 and 31 match at the location of the velocity maximum; this gives at once

$$\int_{\kappa}^{\beta} \left( \frac{\beta^2}{\xi} - \xi \right)^s d\xi = \int_{\beta}^1 \left( \xi - \frac{\beta^2}{\xi} \right)^s d\xi \tag{4.2-32}$$

This relation gives  $\beta$  as a function of the geometrical quantity  $\kappa$  and the power-law exponent  $n$  (see Table 4.2-2).<sup>7</sup>

The volume rate of flow in the annulus is then<sup>7</sup>

$$\begin{aligned} Q &= 2\pi \int_{\kappa R}^R v_z r dr = \pi R^3 \left[ \frac{(\mathcal{P}_0 - \mathcal{P}_L)R}{2mL} \right]^s \int_{\kappa}^1 |\beta^2 - \xi^2|^{s+1} \xi^{-s} d\xi \\ &= \frac{\pi R^3}{(1/n) + 3} \left( \frac{\mathcal{P}_0 - \mathcal{P}_L}{2mL} \right)^{1/n} [(1 - \beta^2)^{1+(1/n)} - \kappa^{1-(1/n)}(\beta^2 - \kappa^2)^{1+(1/n)}] \end{aligned} \tag{4.2-33}$$

This result follows from substituting  $v_z(r)$  from Eqs. 4.2-30 and 31 into the integral for  $Q$ , and then interchanging the order of integration and performing the inner integrals.

TABLE 4.2-2  
Values of  $\beta(\kappa, n)$  Computed from Eq. 4.2-32<sup>a</sup>

$n$	$\kappa$								
	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.10	0.3442	0.4687	0.5632	0.6431	0.7140	0.7788	0.8389	0.8954	0.9489
0.20	0.3682	0.4856	0.5749	0.6509	0.7191	0.7818	0.8404	0.8960	0.9491
0.30	0.3884	0.4991	0.5840	0.6570	0.7229	0.7840	0.8416	0.8965	0.9492
0.40	0.4052	0.5100	0.5912	0.6617	0.7259	0.7858	0.8426	0.8969	0.9493
0.50	0.4193	0.5189	0.5970	0.6655	0.7283	0.7872	0.8433	0.8972	0.9493
0.60	0.4312	0.5262	0.6018	0.6686	0.7303	0.7884	0.8439	0.8975	0.9494
0.70	0.4412	0.5324	0.6059	0.6713	0.7319	0.7893	0.8444	0.8977	0.9495
0.80	0.4498	0.5377	0.6093	0.6735	0.7333	0.7902	0.8449	0.8979	0.9495
0.90	0.4572	0.5422	0.6122	0.6754	0.7345	0.7909	0.8452	0.8980	0.9495
1.00	0.4637	0.5461	0.6147	0.6770	0.7355	0.7915	0.8455	0.8981	0.9496

<sup>a</sup> This table is abstracted from the more complete Table I in R. W. Hanks and K. M. Larsen, *Ind. Eng. Chem. Fundam.*, **18**, 33-35 (1979).

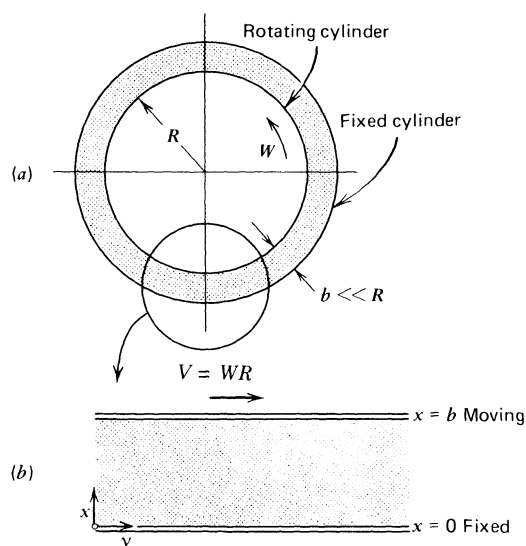


FIGURE 4.2-3. (a) Helical flow in a thin annulus; the fluid flows axially because of a pressure gradient and tangentially because of the rotating inner cylinder. (b) Coordinate system to be used for the equivalent problem neglecting curvature.

In this flow an appreciable portion of the fluid is in a low shear-rate region, namely that which passes through a washer-shaped region about  $r = \beta R$ . The width of this washer-shaped ring increases as the pressure drop decreases, and in that region the viscosity from the power law will be much greater than the experimental value; as a consequence volume flow rates from Eq. 4.2-33 can be expected to be lower than the experimental values. The errors introduced by overestimating the viscosity near  $r = \beta R$  will be negligible if the wall shear stresses at the inner and outer cylinders,  $\tau_{\kappa R}$  and  $\tau_R$ , are both much greater than  $\eta_0 \dot{\gamma}_0$ . This point is pursued further in Example 4.5-2.

#### EXAMPLE 4.2-5 Enhancement of Axial Annular Flow by Rotating Inner Cylinder (Helical Flow of a Power-Law Fluid)<sup>9</sup>

Here we consider the axial flow in a very thin annulus where the inner cylinder radius is  $R$  and the gap width is  $b$ , which is much smaller than  $R$ . The flow is driven by a pressure gradient. We want to investigate the way in which the flow rate changes if the inner cylinder is made to rotate. In this illustrative example we have more than one nonvanishing component of  $\dot{\gamma}$ , so that the flows in the two directions are coupled through the dependence of the non-Newtonian viscosity on the magnitude of  $\dot{\gamma}$  defined in Eq. 4.1-8.

**SOLUTION** The system is sketched in Fig. 4.2-3a. Because of the thinness of the slit, curvature can be neglected and the original problem becomes equivalent to the plane-slit problem shown in Fig. 4.2-3b; in this figure we also show the coordinate system we use. We postulate that  $v_z = v_z(x)$ ,  $v_y = v_y(x)$ ,  $v_x = 0$ , and  $\mathcal{P} = \mathcal{P}(z)$ .

<sup>9</sup> This problem has been studied experimentally and theoretically by A. C. Dierckes, Jr. and W. R. Schowalter, *Ind. Eng. Chem. Fundam.*, **5**, 263-271 (1966); numerical calculations for the Oldroyd model viscosity function (Eq. A of Table 7.3-3) have been made by J. G. Savins and G. C. Wallick, *AIChE J.*, **12**, 357-363 (1966). See also B. D. Coleman, H. Markovitz, and W. Noll, *Viscometric Flows of Non-Newtonian Fluids*, Springer, New York, (1966), pp. 37-41.

For these postulates the equations of motion become:

$$\text{y-component} \quad 0 = -\frac{d}{dx} \tau_{xy} \quad (4.2-34)$$

$$\text{z-component} \quad 0 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{L} - \frac{d}{dx} \tau_{xz} \quad (4.2-35)$$

where  $\mathcal{P}_0 - \mathcal{P}_L$  is the modified pressure drop between  $z = 0$  and  $z = L$ . The components of the stress tensor for the power-law fluid are

$$\tau_{xy} = -\eta(\dot{\gamma})\dot{\gamma}_{xy} = -m\dot{\gamma}^{n-1}\dot{\gamma}_{xy} \quad (4.2-36)$$

$$\tau_{xz} = -\eta(\dot{\gamma})\dot{\gamma}_{xz} = -m\dot{\gamma}^{n-1}\dot{\gamma}_{xz} \quad (4.2-37)$$

in which the magnitude of the rate-of-strain tensor is

$$\dot{\gamma} = \sqrt{\left(\frac{dv_y}{dx}\right)^2 + \left(\frac{dv_z}{dx}\right)^2} \quad (4.2-38)$$

It is convenient to use dimensionless quantities

$$\bar{x} = \frac{x}{b}; \quad \bar{v}_i = \frac{v_i}{V}; \quad a = \frac{(\mathcal{P}_0 - \mathcal{P}_L) b^{n+1}}{mL V^n} \quad (4.2-39)$$

and further to abbreviate  $d\bar{v}_y/d\bar{x}$  by  $Y$  and  $d\bar{v}_z/d\bar{x}$  by  $Z$ . Then the equations of motion, in terms of the velocity gradients, become

$$\frac{d}{d\bar{x}} [(Y^2 + Z^2)^{(n-1)/2} Y] = 0 \quad (4.2-40)$$

$$\frac{d}{d\bar{x}} [(Y^2 + Z^2)^{(n-1)/2} Z] = -a \quad (4.2-41)$$

These equations may be integrated at once, and we use the symbols  $C_1$  and  $C_2$  for the constants of integration that appear on the right side. When these equations are then solved for  $Y$  and  $Z$ , we get

$$Y = \frac{d\bar{v}_y}{d\bar{x}} = C_1 [C_1^2 + (C_2 - a\bar{x})^2]^{(1-n)/2n} \quad (4.2-42)$$

$$Z = \frac{d\bar{v}_z}{d\bar{x}} = (C_2 - a\bar{x}) [C_1^2 + (C_2 - a\bar{x})^2]^{(1-n)/2n} \quad (4.2-43)$$

From this point on we specialize to  $n = 1/3$ , since for this choice an analytical solution can be obtained. The data in §4.1 suggest that  $n = 1/3$  is very nearly appropriate for some polymer solutions.

Integration of Eqs. 4.2-42 and 4.2-43 with  $n = 1/3$  gives

$$\bar{v}_y = \int_0^{\bar{x}} C_1 [C_1^2 + (C_2 - a\bar{x})^2] d\bar{x} \quad (4.2-44)$$

$$\bar{v}_z = \int_0^{\bar{x}} (C_2 - a\bar{x}) [C_1^2 + (C_2 - a\bar{x})^2] d\bar{x} \quad (4.2-45)$$

in which the integration constants have been set equal to zero since both velocity components are zero at  $\bar{x} = 0$ . The boundary condition that  $d\bar{v}_z/d\bar{x} = 0$  at  $\bar{x} = 1/2$  then leads to

$$C_2 = \frac{a}{2} \quad (4.2-46)$$

The boundary condition that  $\bar{v}_y = 1$  at  $\bar{x} = 1$  leads to the cubic equation  $C_1^3 + \frac{1}{12}a^2C_1 - 1 = 0$ , which has only one real root, according to Descartes' rule of signs. That root is

$$C_1 = A_+ + A_- \quad (4.2-47)$$

in which

$$A_{\pm} = \sqrt[3]{\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{27}\left(\frac{a^2}{12}\right)^3}} \quad (4.2-48)$$

For large values of  $a$  (i.e., large pressure drop or small angular velocity of the inner cylinder) this last expression can be expanded as

$$C_1 = (12/a^2) - (12/a^2)^4 + \dots \quad (4.2-49)$$

This expansion is used presently.

The integrals in Eqs. 4.2-44 and 45 are now performed to give the velocity profiles (see Fig. 4.2-4):

$$\bar{v}_y = C_1 \left[ C_1^2 \bar{x} + \frac{a^2}{12} (3\bar{x} - 6\bar{x}^2 + 4\bar{x}^3) \right] \quad (4.2-50)$$

$$\bar{v}_z = \frac{a}{2} \left[ \left( C_1^2 + \frac{a^2}{4} \right) \bar{x} - \frac{1}{2} a^2 \bar{x}^2 + \frac{1}{3} a^2 \bar{x}^3 \right] - a \left[ \left( C_1^2 + \frac{a^2}{4} \right) \frac{\bar{x}^2}{2} - \frac{1}{3} a^2 \bar{x}^3 + \frac{1}{4} a^2 \bar{x}^4 \right] \quad (4.2-51)$$

and the axial volume rate of flow through the annulus is

$$\begin{aligned} Q &= 2\pi R \int_0^b v_z dx = 2\pi R^2 b W \int_0^1 \bar{v}_z d\bar{x} \\ &= \frac{\pi R^2 b W a^3}{40} \left[ 1 + \frac{20}{3} \left( \frac{C_1}{a} \right)^2 \right] \end{aligned} \quad (4.2-52)$$

in which  $a = (b\Delta\mathcal{P}/mL)(b WR)^{1/3}$ . Use of Eq. 4.2-49 then leads to

$$\begin{aligned} Q &= \frac{\pi R^2 b W a^3}{40} \left[ 1 + \frac{960}{a^6} + \dots \right] \\ &= \frac{\pi R b^2}{40} \left( \frac{b\Delta\mathcal{P}}{mL} \right)^3 \left[ 1 + 960 \left( \frac{WR}{b} \right)^2 \left( \frac{mL}{b\Delta\mathcal{P}} \right)^6 + \dots \right] \end{aligned} \quad (4.2-53)$$

in which  $\Delta\mathcal{P} = \mathcal{P}_0 - \mathcal{P}_L$ . This shows that the flow in the axial direction is enhanced because of the imposed shearing in the tangential direction, since this additional shearing causes the viscosity to be lowered. Note that the correction term is very sensitive to the slit width, which enters as the inverse eighth power, and the pressure gradient, which appears to the minus sixth power. This is a good

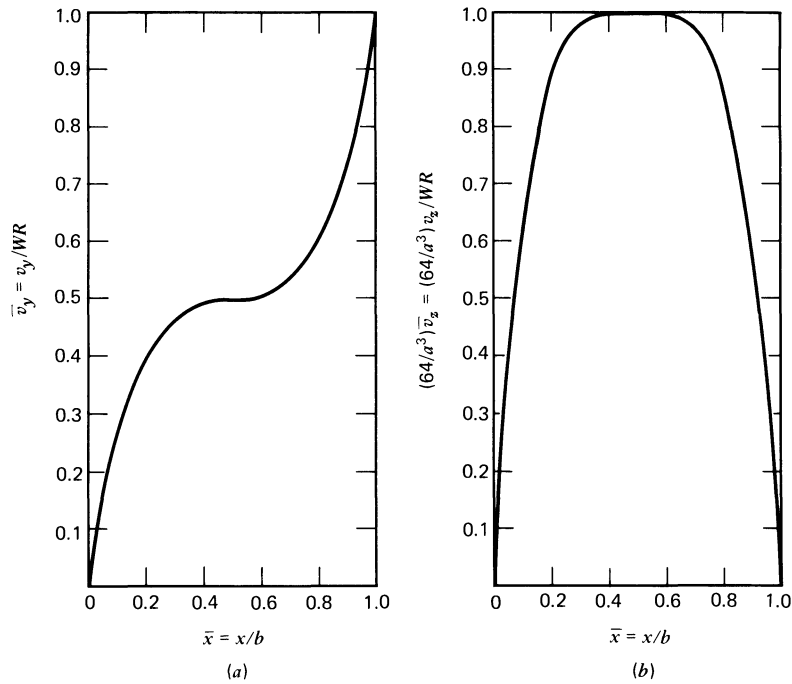


FIGURE 4.2-4. Velocity profiles for helical flow in a thin annular gap. (a) The dimensionless tangential velocity  $\bar{v}_y = v_y/WR$ , given by Eq. 4.2-50, and (b) the dimensionless axial velocity  $\bar{v}_z = v_z/WR$ , given by Eq. 4.2-51. In these velocity expressions  $C_1$  has been taken to be  $12/a^2$  (see Eq. 4.2-49). If there were no axial flow,  $\bar{v}_y$  would be the linear function  $\bar{v}_y = \bar{x}$ ; the deviation from linearity results from the change in viscosity across the cross section, brought about by the axial flow.

illustration of how the power-law model can be used to ascertain the sensitivity of a quantity (e.g.,  $Q$ ) to the key parameters in the system (e.g.,  $b$  and  $\Delta\mathcal{P}$ ).

**EXAMPLE 4.2-6** Flow Enhancement Produced by a Pulsatile Pressure Drop in a Circular Tube (Quasi-Steady-State Approximation)<sup>10</sup>

A polymer is flowing axially in a horizontal circular tube of radius  $R$  and length  $L$  as a result of a sinusoidally varying pressure drop

$$\Delta p = \Delta p_0 \cdot (1 + \varepsilon \Re e\{e^{i\omega t}\}) \tag{4.2-54}$$

in which  $\Delta p = p_0 - p_L$ , and  $\Delta p_0$  is the corresponding quantity for  $\varepsilon = 0$ . The parameter  $\varepsilon$  is presumed to be small with respect to unity.

a. Find the volume rate of flow  $Q(t)$  by applying Eq. B of Table 4.2-1 instantaneously. Then find  $\langle Q \rangle$ , the average value of  $Q$  over one cycle of oscillation; also find the flow-rate enhancement  $(\langle Q \rangle - Q_0)/Q_0$ , where  $Q_0$  is the flow rate with  $\varepsilon = 0$ .

b. Find the power requirement for pumping the material through the circular tube with a pulsatile pressure gradient. What conclusions can you draw?

<sup>10</sup> H. A. Barnes, P. Townsend, and K. Walters, *Rheol. Acta*, **10**, 517-527 (1971); *Nature*, **224**, 585-587 (1969). See also J. M. Davies, S. Bhumiratana, and R. B. Bird, *J. Non-Newtonian Fluid Mech.*, **3**, 237-259 (1977/1978), and N. Phan-Thien and J. Dudek, *J. Non-Newtonian Fluid Mech.*, **11**, 147-161 (1982). Viscoelastic effects are discussed in Example 7.4-2.

**SOLUTION** (a) In the quasi-steady-state approximation

$$Q(t) = \frac{\pi R^3}{(1/n) + 3} \left( \frac{\Delta p_0 R}{2mL} \right)^{1/n} (1 + \varepsilon \Re\{e^{i\omega t}\})^{1/n} \quad (4.2-55)$$

for the power-law fluid model.

Then the time average volume flow rate is

$$\begin{aligned} \frac{\langle Q \rangle}{Q_0} &= \langle (1 + \varepsilon \Re\{e^{i\omega t}\})^{1/n} \rangle \\ &= 1 + \varepsilon \frac{1}{n} \langle \Re\{e^{i\omega t}\} \rangle + \varepsilon^2 \frac{1}{2} \left( \frac{1}{n} \right) \left( \frac{1}{n} - 1 \right) \langle [\Re\{e^{i\omega t}\}]^2 \rangle + \dots \end{aligned} \quad (4.2-56)$$

in which an expansion for small  $\varepsilon$  has been made. The time average of  $\Re\{e^{i\omega t}\}$  is zero, and the time average of its square is<sup>11</sup>

$$\langle [\Re\{e^{i\omega t}\}]^2 \rangle = \frac{1}{2} \langle \Re\{e^{2i\omega t}\} + 1 \rangle = \frac{1}{2} \quad (4.2-57)$$

The flow-rate enhancement is then given by:

$$\frac{\langle Q \rangle - Q_0}{Q_0} = \frac{1}{4} \left( \frac{1-n}{n^2} \right) \varepsilon^2 + O(\varepsilon^4) \quad (4.2-58)$$

The power law thus predicts that the enhancement increases as  $n$  decreases from unity (i.e., as the fluid becomes more non-Newtonian), and that it is independent of the frequency.

(b) The time-averaged power  $\langle P \rangle$  required to pump the fluid is:<sup>12</sup>

$$\begin{aligned} \langle P \rangle &= \left\langle \iint v_z \Delta p \, r dr d\theta \right\rangle \\ &= \langle Q \Delta p \rangle \\ &= Q_0 \Delta p_0 \langle (1 + \varepsilon \Re\{e^{i\omega t}\})^{(1/n)+1} \rangle \\ &= Q_0 \Delta p_0 \left( 1 + \varepsilon^2 \frac{1}{2} \left( \frac{1}{n} + 1 \right) \left( \frac{1}{n} \right) \langle [\Re\{e^{i\omega t}\}]^2 \rangle + \dots \right) \end{aligned} \quad (4.2-59)$$

Hence the fractional increase in power needed is

$$\frac{\langle P \rangle - P_0}{P_0} = \frac{1}{4} \left( \frac{1+n}{n^2} \right) \varepsilon^2 + \dots \quad (4.2-60)$$

<sup>11</sup> The following relation is useful:

$$\Re\{w_1\} \Re\{w_2\} = \frac{1}{2} [\Re\{w_1 w_2\} + \Re\{w_1 \bar{w}_2\}] \quad (4.2-56a)$$

in which  $w_1$  and  $w_2$  are two complex quantities and the overbar indicates a complex conjugate.

<sup>12</sup> S. Middleman, *Fundamentals of Polymer Processing*, McGraw-Hill, New York (1977), p. 114.

Comparison of Eqs. 4.2-58 and 60 shows that the flow-rate enhancement increases less rapidly than the power requirement. Hence there is no energetic advantage to pumping with a sinusoidal pressure gradient.

Flow enhancement under sinusoidal pumping has been observed experimentally; data comparisons are given in Example 7.4-2, where this problem is solved again using a constitutive equation that describes elastic effects, and without the quasi-steady-state assumption. It is found there that the enhancement in Eq. 4.2-58 is exact through order  $\varepsilon^2$  for the power-law viscosity function.

**EXAMPLE 4.2-7** Squeezing Flow between Two Parallel Circular Disks (Lubrication Approximation and Quasi-Steady-State Approximation)

Analyze the flow of a power-law fluid in the gap between two circular disks that approach one another according to some prescribed velocity (see Fig. 4.2-5). The velocity of the upper plate is given by  $\dot{h} = dh/dt$ . Use a lubrication approximation and a quasi-steady-state assumption; that is, assume that the instantaneous volume rate of flow  $Q(r)$  across the cylindrical surface at  $r$  is that for flow through a slit of thickness  $2h$  and width  $2\pi r$ . Equate this  $Q(r)$  to the volumetric flow rate obtained by a conservation-of-mass statement.

Obtain the time required to squeeze out half of the liquid in the gap by the application of a constant force  $F$  on the disks.

**SOLUTION** Conservation of mass states that, for an incompressible fluid, the volume rate at which fluid crosses the cylindrical surface at  $r$  should equal the rate at which the volume between the two plates within the cylindrical surface at  $r$  decreases:

$$Q(r) = 2\pi r^2 (-\dot{h}) \tag{4.2-61}$$

To apply Eq. A of Table 4.2-1 to the flow between the disks in the region between  $r$  and  $r + dr$  we make the following changes of notation:

$$\begin{aligned} W &\rightarrow 2\pi r \\ B &\rightarrow h \\ (\mathcal{P}_0 - \mathcal{P}_L)/L &\rightarrow -dp/dr \\ Q &\rightarrow Q(r) \end{aligned}$$

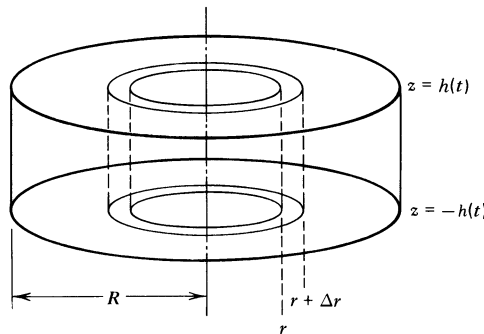


FIGURE 4.2-5. Squeezing flow between two circular disks of radius  $R$ . The instantaneous disk separation is  $2h(t)$ .

Then we get for the volume rate of flow:

$$Q(r) = \frac{2 \cdot 2\pi r \cdot h^2}{(1/n) + 2} \left( -\frac{h}{m} \frac{dp}{dr} \right)^{1/n} \quad (4.2-62)$$

When Eqs. 4.2-61 and 62 are equated, we get an equation for  $p$  as a function of  $r$ . When this is integrated with respect to  $r$ , with the boundary condition  $p = p_a$  at  $r = R$ , we get

$$p - p_a = m \frac{(-\dot{h})^n}{h^{2n+1}} \left( \frac{2n+1}{2n} \right)^n \frac{R^{n+1}}{n+1} \left[ 1 - \left( \frac{r}{R} \right)^{n+1} \right] \quad (4.2-63)$$

where  $p_a$  is the atmospheric pressure.

The force on the upper plate required to move the plate at a speed  $\dot{h}$  is then:

$$F(t) = \int_0^{2\pi} \int_0^R (p - p_a + \tau_{zz}) \Big|_{z=h} r dr d\theta \quad (4.2-64)$$

The normal stress  $\tau_{zz}$  at the upper plate is zero by the same arguments used for Newtonian fluids (see Example 1.2-1). When  $p - p_a$  from Eq. 4.2-63 is substituted into the integral for  $F(t)$  we get

$$F(t) = \frac{(-\dot{h})^n}{h^{2n+1}} \left( \frac{2n+1}{2n} \right)^n \frac{\pi m R^{n+3}}{n+3} \quad (4.2-65)$$

This is the *Scott equation*,<sup>13,14</sup> which was first developed for measuring the non-Newtonian viscosity of unvulcanized rubber stocks.

For a constant force  $F$ , Eq. 4.2-65 is an ordinary differential equation for  $h(t)$ . When this equation is integrated from  $t = 0$  (when  $h = h_0$ ) to  $t = t_{1/2}$  (when  $h = h_0/2$ ), the following equation is obtained for  $t_{1/2}$ :

$$\frac{t_{1/2}}{n} = K_n \left( \frac{\pi R^2 m}{F} \right)^{1/n} \left( \frac{R}{h_0} \right)^{1+(1/n)} \quad (4.2-66)$$

in which  $K_n$  is a constant that depends only on  $n$

$$K_n = \left( \frac{2^{1+(1/n)} - 1}{2n} \right) \left( \frac{2n+1}{n+1} \right) \left( \frac{1}{n+3} \right)^{1/n} \quad (4.2-67)$$

Equation 4.2-66 has been tested by Leider<sup>15</sup> who measured  $t_{1/2}$  and  $F$  in 181 experimental runs. His data are compared with the power-law result in Fig. 4.2-6, where suitably chosen dimensionless groups are used. The characteristic time constant  $\lambda$  for the fluid is defined as

$$\lambda = (m'/2m)^{1/(n'-n)} \quad (4.2-68)$$

<sup>13</sup> J. R. Scott, *Trans. Inst. Rubber Ind.*, **7**, 169-186 (1931); **10**, 481-493 (1935).

<sup>14</sup> S. Oka in F. R. Eirich, ed., *Rheology*, Vol. 3, Academic Press, New York (1960), Chapt. 2, pp. 73-75; A. Cameron, *The Principles of Lubrication*, Longmans, Green, and Co., London (1966), pp. 389-392; D. F. Moore, *The Friction and Lubrication of Elastomers*, Pergamon, Elmsford, NY (1972); A. B. Metzner, *Rheol. Acta*, **10**, 434-444 (1971); M. L. DeMartine and E. L. Cussler, *J. Pharm. Sci.*, **64**, 976-982 (1975); G. Brindley, J. M. Davies, and K. Walters, *J. Non-Newtonian Fluid Mech.*, **1**, 19-37 (1976).

<sup>15</sup> P. J. Leider, *Rheology Research Center Report No. 22*, Nov. 1973, University of Wisconsin, Madison; *Ind. Eng. Chem. Fundam.*, **13**, 342-346 (1974). Additional experimental testing of the squeeze-flow equations and further examination of the theory have been carried out by R. J. Grimm, *AIChE J.*, **24**, 427-439 (1978).

THE GENERALIZED NEWTONIAN FLUID

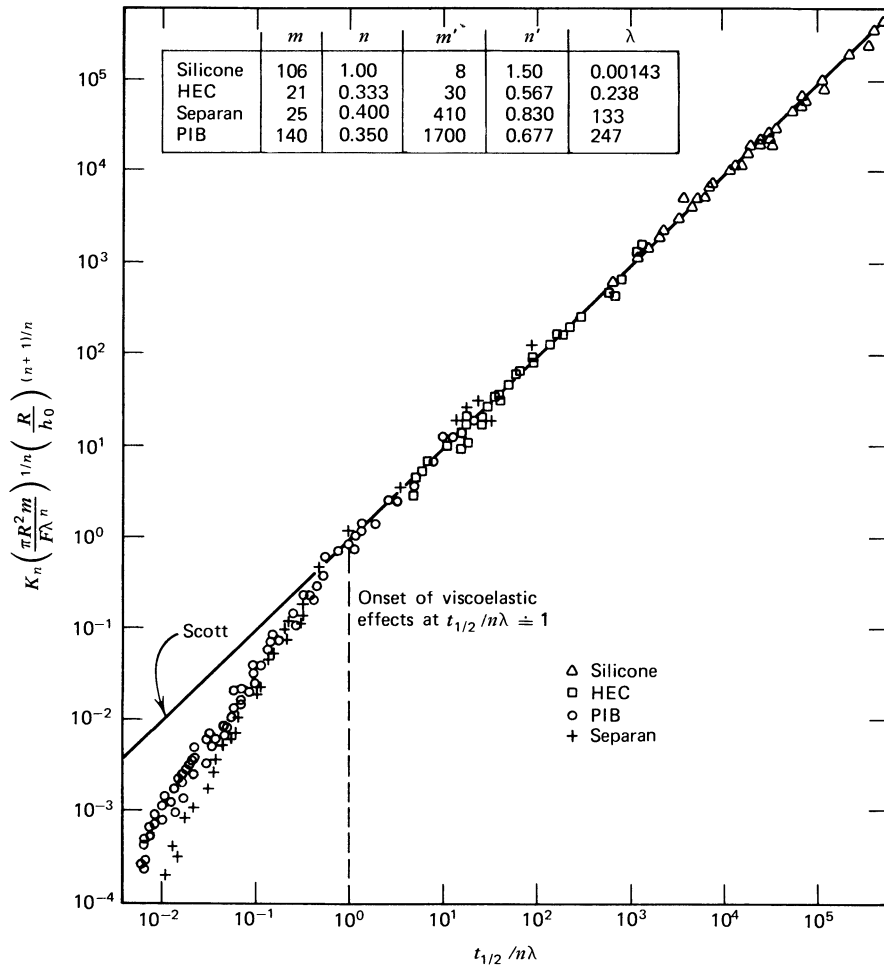


FIGURE 4.2-6. Squeeze flow data of P. J. Leider (University of Wisconsin Rheology Research Center Rept. No. 22, Nov. 1973) along with the Scott equation (Eq. 4.2-66). The fluid parameters for Eq. 4.2-68 are shown in the upper left-hand corner;  $m$  has units of  $\text{Pa} \cdot \text{s}$ ,  $m'$  has units of  $\text{Pa} \cdot \text{s}'$ , and  $\lambda$  is given in seconds. [Reprinted with permission from P. J. Leider, *Ind. Eng. Chem. Fundam.*, **13**, 342-346 (1974). Copyright by the American Chemical Society.]

in which  $m$  and  $n$  are the constants in the power law ( $\eta = m\dot{\gamma}^{n-1}$ ) and  $m'$  and  $n'$  are constants describing the power-law region of the first normal stress coefficient ( $\Psi_1 = m'\dot{\gamma}^{n'-2}$ ). It is seen that for the fluids tested the power-law model describes the data down to  $t_{1/2}/n\lambda = 1$ , but below that value there are marked deviations from the power law because of elastic effects. This point is discussed further in §4.6. The success of the Scott equation indicates that the squeeze-flow experiment may be useful for determining fluid parameters from measurements of  $F$  and  $t_{1/2}$ . From Eq. 4.2-66 we see that a log-log plot of  $(Fh_0/R^3)$  vs.  $(R/h_0 t_{1/2})$  will, for large  $t_{1/2}$ , give  $n$  from the slope of the straight line and  $m$  from an intercept

$$\log \left( \frac{Fh_0}{R^3} \right) = \log [\pi m (n K_n)^n] + n \log \left( \frac{R}{h_0 t_{1/2}} \right) \quad (4.2-69)$$

From a log-log plot of  $(R^2/F)^{1/n} (R/h_0)^{1+(1/n)}$  vs.  $t_{1/2}/n$  one can get  $\lambda$  from the break in the curve at  $t_{1/2}/n = \lambda$ . Explicit illustrations have been given by Leider.<sup>15</sup>

## 4B.15 Development of Design Equation for Manifold of a “Coat-Hanger” Die (Power Law)<sup>23</sup>

Plastic sheeting can be made by extruding the molten polymer through a “coat-hanger” die made up of an entry tube, two manifolds, and a slit (see Fig. 4B.15). The manifold is a tube of circular cross section, whose radius  $\bar{R}$  varies in the direction  $\bar{z}$  of the manifold axis. Our object is to design the manifold (i.e., find  $\bar{R}(\bar{z})$ ) so that the flow through the slit will be uniform; that is the volume flow rate of the slit must not vary in the  $x$ -direction.

The slit has a total width  $2W$  and has a thickness  $2B$ . The volume rate of flow into the entry tube is  $2Q_0$ , with half of the fluid going into the left manifold and half into the right manifold.

a. Consider a width  $\Delta x$  of the slit. What is the volume flow rate through this portion of the slit?

b. Make a mass balance over a length  $\Delta \bar{z}$  of the manifold tube and then let  $\Delta \bar{z}$  go to zero to get the differential equation:

$$-\frac{d\bar{Q}}{d\bar{z}} = \frac{Q_0}{W} \cos \alpha \quad (4B.15-1)$$

where  $\bar{Q}$  is the volume flow rate at  $\bar{z}$ . Draw a carefully labelled diagram to show how you derive this relation.

c. Let  $\bar{p}(\bar{z})$  be the pressure as a function of  $\bar{z}$  in the manifold. Let  $p(z)$  be the pressure in the slit. Why is  $p(-L(x)) \doteq \bar{p}(\bar{z})$ ? How are  $x$  and  $\bar{z}$  related? How are  $W$  and  $\bar{L}$  related?

<sup>23</sup> J. R. A. Pearson, *Trans. J. Plast. Inst.*, **32**, 239 (1964); J. R. A. Pearson, *Mechanics of Polymer Processing*, Elsevier Applied Science, New York (1985), §10.2. See also Z. Tadmor and C. G. Gogos, *Principles of Polymer Processing*, Wiley, New York (1979), pp. 545–551.